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The So-Called "Transformation Problem" Revisited

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Much ink has been spilled since von Bortkiewicz first criticized the way Marx dealt with the problem of transforming values into prices of production in Volume III of *Capital* [12]. Although no one any longer pretends that this discovery marks the beginning of the "crisis of marxism" (so technical a question deserves neither the credit nor the blame for that), it remains true that the solution most widely accepted at the moment (which is expressed most completely in the work of M. Morishima[1]) greatly weakens the thrust of Marx's theory of value and surplus value. The accepted approach reduces that theory at the most to a rough and approximate expression of the idea that workers do not receive the whole fruits of their labor in the wage. This at any rate is the conclusion of Morishima himself in his "Fundamental Marxian Theorem" ("the equilibrium rate of profit is positive if and only if the rate of exploitation is positive"), and of Samuelson, who says

> Although *Capital*'s total findings need not have been developed in dependence upon Volume I's digression into surplus values, its essential insight does depend crucially on comparison of the subsistence goods needed to produce and reproduce labor with what the undiluted labor theory of value calculates to be the amount of goods producible for all classes in view of the embodied labor requirements of the goods. The tools of bourgeois analysis could have been used to discover and expound this notion of exploitation if only those economists had been motivated to use the tools for this purpose. [18, p. 421]

What Samuelson means by "tools of bourgeois analysis" is the theory that the time by which an owner of money agrees to defer his consumption is as scarce a good as labor, and deserves its own special income (interest).

* Translated into English by Duncan K. Foley with the assistance of the author.

[1] See especially his "Marx's Economics" [13]. Earlier work in this line includes that of Medio, Meek, Okishio, Seton, Sweezy, Roubine, Winternitz, and others. All those mentioned treat the question along the same general lines as Morishima. For a survey of this literature, from Marx to von Bortkiewicz, see Dostaler [5]; from von Bortkiewicz to the present, see Samuelson [18] and Benetti *et al.* [3]. Morishima and Catephores have proposed a new solution in terms of Markov processes [14]. But this work adopts the same approach as the earlier work on the essential question which will be developed here, the treatment of variable capital.

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In this situation several Marxist writers take refuge in a refusal to adopt formalisations of Morishima's kind, putting forward critical observations which are coherent, but do not address this issue directly (Salama [17] and Yaffé [18]) or even refuse, in the name of doubtful epistemological principles, to confront the problem of the transformation at all (Benetti *et al.* [3]).

I propose to argue here that solutions of Morishima's kind, once they are filled out and correctly understood, do not contradict any of Marx's aims in *Capital*, but that there exists another solution,[2] closer to the approach of *Capital*, which exhibits the famous equalities of Volume III, between the sum of prices and the sum of values, and the sum of profits and the sum of surplus value, equalities which are inconsistent with Morishima's treatment. Moreover, it will be shown that the rate of profit in this treatment depends on the rate of exploitation and the structure of output, rather than on the workers' consumption bundle, as in Morishima's analysis.

The first part of the paper will review Marx's treatment of the problem, and the critiques of it; the second part will treat Morishima's solution; the third will put forward the new solution; and the last part will compare the properties of the two solutions.

I. Marx's Solution and Its Limitations

For Marx,[3] the commodity character of the economy confers on the products of economic agents a value proportional to the fraction of the social labor allocated to their production and validated by society in exchange. To adopt modern terminology, one could say that to each bundle of commodities, represented by a vector $\mathscr{y}$ in the space having for its natural basis the n different units of use-value, value, a linear form $\mathscr{v}$ on that space, associates a positive number.[4]

In an assumed representative productive process, the "living" labor applied to means of production adds value to the value already created by

[2] Dumenil [7] laid the foundations of this solution. I would like to thank him for helpful conversations on this problem, though my approach is rather different from his. At the same time, but completely independently, Foley [8] proposed a substantially similar solution.

[3] The extremely brief summary which follows raises many problems which are discussed in relation to Marx's text in my book [9]. Here I confine myself to an examination of the formalism used in studying the "transformation problem."

[4] Vectors $\mathscr{y}$ (and linear forms, or covectors $\mathscr{v}$) will be denoted by script and are treated as column vectors in right-hand side operations (and row vectors in left-hand side operations). According to the Einstein convention, indices are superscript for vectors (and subscript for covectors). Scalars and matrices will be denoted by italics.

"dead" labor incorporated in the means of production.[5] Then, if a_j^i is the quantity of good i normally required to produce one unit of good j, and if A is the corresponding matrix of these production coefficients, we have $v = vA + \ell$, where $\ell = l_1, \ldots, l_n$ is the vector of quantities of abstract labor incorporated concretely in units of the goods. As a result:

$$v = \ell(I - A)^{-1}.$$

This procedure of "adding" current labor to the value already embodied in the goods[6] has been criticised, and some French marxists refuse to adopt it (see [4, Sect. 24]). We can show nevertheless that it is legitimate, and inherent in the form of value, as long as the techniques of production remain the same, and we are analyzing the reproduction of the economic system. Under these assumptions the past labor expended in one branch of production corresponds to the current labor expended in another branch. Of course, this no longer holds if we permit changes in technique, which are part of the causes of crisis and inflation (see [9]). In any case, the debate on the "transformation problem" always (though rarely explicitly) makes these assumptions.

So far we have had nothing to say about the capitalist character of the economy. The marxist approach, like all scientific approaches, proceeds by successive approximations: We first study bodies falling in a vacuum, then introduce the resistance of the atmosphere, magnetic fields, etc. In the first section of Volume I of *Capital*, Marx enunciates and analyzes the law of value (its substance, form, quantity) "in general," that is, for any commodity economy. He then analyzes the "divisions" of the flow of value which correspond to capitalist social relations (between constant capital C, variable capital V, and surplus value S), and finally, in Volume III, the modifications induced in the law of value itself by those relations.

In capitalist social relations labor power appears (to the capitalist, not to the worker, of course) as a commodity. This commodity has a value w: the amount of abstract labor which workers have the "right," given the historical stage of development, to spend on the market to reproduce their labor power from day to day. It also has a use value: the ability to produce abstract labor, and hence to add value to commodities. The amount of abstract labor

[5] By "representative productive process" I mean a process operating with current standard techniques. Such a process is characterized by a technique which gives the quantities of means of production and labor required, and the time of labor taken by the technique. See Aglietta [1] and Lipietz [9].

[6] Notice that v and ℓ are measured in the same units, abstract labor time. Once v is computed, we could renormalize the matrix A by choosing as the unit of each good the quantity of the good which has unit value. The coefficients a_j^i would then be pure numbers, rather than quantities of good i per unit of good j.

va (value added) that can be extracted from this commodity is determined by the duration, λ, and the intensity, ε, of labor. The three parameters w, λ, ε are the results of a historical process, of the "class struggle" (see Marx [12, Vol. I, Chap. X]). Together they determine the rate of surplus value, or rate of exploitation, e, which is the ratio of the "unpaid labor," $va - w$, to the value of the labor power w. Thus by definition:

$$w(1 + e) = 1.$$

In modern vector terminology, we have

$$v = vA + w\ell + ew\ell.$$

This is the modern form of "$C + V + S$." Notice that in adopting this notation we have assumed that the quantity of the commodity labor power which it is necessary to buy and put in motion to produce the good j is measured by the same number as the quantity of abstract labor produced by that labor power. That is, we have assumed that the duration and intensity of work are given, and that we have taken as our unit of labor power the day (for example), and for our unit of value the abstract labor expended in a day. The techniques of production, of course, are taken to be fixed socially, and concrete labor is counted directly as social abstract labor. To put it another way, we assume the existence of a linear transformation T, mapping n-tuples of the commodity "labor power" into covectors of "value added." This transformation, the coefficients of which are defined by the intensity and duration of work, could be called the "tensor of exploitation" (see Appendix A). These may appear to be relatively minor points, but if we forget them we are in danger of identifying, along with the pre-Marxist classical economists (Smith, Ricardo), "labor commanded" and "labor embodied,"[7] and, what is very important for our discussion, of confusing a social relation with a technical relation between certain input–output coefficients.

The illusion of a purely technical relation between inputs and outputs is completed once we suppose, as we are entitled to do, that, just as the production of good j is characterized by the given coefficients of the representative process (a_j^i, l_j), so there exists a vector of normal consumption per unit of labor time, d. We then have

$$w = v \cdot d$$

[7] The Benetti and Cartelier criticism [3] of Morishima-type solutions turns on this point. Still, instead of positing the existence of this tensor, these writers refuse point blank to confront the problem of the relation between the space of values and the space of relative prices.

and the existence of the autonomous commodity "labor power" necessary to the production of a unit of commodity j, l_j, is absorbed by the given bundle of commodities necessary indirectly for the production of a unit of j, the vector $(d^i l_j)$, which must be added to the vector (a^i_j). Then we get a "social–technical" matrix by adding to A the tensor product (or Kronecker product) $d \times \ell$:

$$M = A + d \times \ell,$$

which is only an abbreviation for:

$$M = A + d \times (\ell T^{-1}),$$

the linear transformation T being reduced to the identity matrix by the choice of units. This matrix looks purely "technical," but the three determinant elements of the labor theory of value and exploitation are already incorporated in it. d expresses the value of labor power, once it is multiplied by v; and T the relation between the quantity of labor power and the quantity of labor embodied in the commodities (which depends on the duration and intensity of labor).

But this is only the beginning of the development of the law of value. If the exchange of products is regulated through competition on the average by the value of products in a pure exchange economy, what regulates exchanges between products of economic agents which are capitalists, in an economy where "commodities are exchanged as products of capital" [12, Vol. III, p. 175], that is, products of labor hired by capital, not of labor alone? The answer, according to Marx, is a transformed value, in which the quantity of surplus value (now called profit) received by each capitalist is proportional to the capital invested. The value produced in a period by society as a conserved quantity finds itself reallocated (again, by competition, but by the competition of capitals [12, Vol. III, p. 180]) among the products, the "transformed" values, and their expression in money, the "prices of production," being fixed so that the surplus value (that is, the part of the produced value which does not return to the direct producers) is distributed among the capitalists in proportion to the capital they have invested. As a result the sum of the prices of production will be equal to the sum of the values, and the sum of the profits will be equal to the sum of surplus value, or at least the ratio of these two pairs of quantities will be the same (the ratio depending on the choice of the numeraire).

Is this mathematically possible? Marx thought so, since this result is a direct expression of his theory of value and of exploitation. And he devised in Volume III of *Capital* (of which he left only a draft at his death) a simple algorithm.

Suppose that we divide the economy into branches, each producing a value: $M_i = C_i + V_i + S_i$. (In the earlier notation, M represents the quantity of value $v_i y^i$, where y_i is the quantity of the good produced in sector i.) In each branch the capital invested has a value $C_i + V_i$ (assuming that the rate of turnover is unity). The total capital engaged is $\Sigma(C_i + V_i)$. The total surplus value is ΣS_i. The general rate of profit is $\Sigma(S_i)/\Sigma(C_i + V_i) = r$. If every branch must realise the same rate of profit, then we need only redistribute the surplus value, and we have the price of production in each sector:

$$PP_i = (C_i + V_i)(1 + r).$$

Clearly the sum of these prices of production will be equal to the sum of the values, and the sum of the profits at these prices will be the same as the sum of the surplus value. The average rate of profit is determined by the rate of surplus value, e; the organic composition of capital in the various sectors, C_i/V_i; and the distribution of the total social capital in these different sectors, and hence by the vector of outputs, y [12, Vol. III, p. 163].

This model has two limitations. First, it assumes that all the sectors have the same period of production. Marx worked out long calculations to investigate the effect of weakening this assumption. But that assumption is not the focus of the controversy. In theory it is always legitimate to bring in complications one by one. We will proceed this way ourselves, in this article.[8] The discussion has centered on another assumption, this one less defensible. Capitalists do not buy the elements of constant capital and variable capital at their values, but at their prices of production. The redistribution of value affects not only S, but also C and V. Marx himself notes this, (e.g., [12, Vol. III, p. 165]), but judges that "our present analysis does not necessitate a closer examination of this point," and passes on to subjects which he judges to be more important. What a mistake! People won't forgive the small slips of great men. And, worse yet, those who have undertaken to correct this slip have proceeded in a path which denies the basic point of Marx's argument: that profit, far from being a compensation for waiting paid to voluntary saving, is only an unpaid part of the value created by the productive workers.

II. The Accepted Solution to the Transformation Problem

To correct Marx's slip, we must assume that the elements of constant and variable capital are bought at their prices of production. For the elements of

[8] On the introduction of fixed capital, see [13] and many other sources. In the same spirit, I will always assume that the matrices are indecomposable.

constant capital this is not a great difficulty: It is necessary only to evaluate the inputs at their prices of production. But what does it mean to say that variable capital is purchased at its price of production? This is a complex issue on its face. Variable capital is, properly speaking, only a quantity of money paid to workers, which represents a fraction (determined by the rate of exploitation) of the value added. Those who first put forward solutions to the problem hit upon an idea that, as it turns out, is not at all neutral. They identified the value of labor power, not with this fraction of the value added, but with the value of the goods d which the money would buy if all the workers adopted the same pattern of consumption and spent their money in a market where prices were proportional to the system of labor values.

Let us follow this path for a moment. As a consequence, we must treat variable capital like constant capital, and evaluate the bundle d at the same prices of productions. Suppose p is the covector of prices, and r the average rate of profit, if it exists. Then:

$$p = (pA + (p \cdot d)\ell)(1 + r),$$
$$(1/(1 + r))\, p = p(A + d \times \ell) = pM.$$

p is thus the eigenvector corresponding to the eigenvalue $(1/(1 + r))$ of the social–technical matrix M.

p is semi-positive, as is M. The Perron–Frobenius theorem (see Nikaido, [15]) tells us that p must be the eigenvector associated with the largest eigenvalue $\mu(M)$. We have, then,

$$r = 1/\mu(M) - 1.$$

First of all, notice that μ depends only on M, hence on A, ℓ, and d, which also determine e. Among the set of vectors d such that $v \cdot d = w$ (that is, for a constant rate of surplus value), the rate of profit r depends on the composition of d, that is, on workers' consumption, and not at all on the structure of output y, that is, not on the "distribution of capital in the different spheres."

But there is worse to come. Suppose we choose the numeraire so that the sum of prices equals the sum of values, that is, $v \cdot y = p \cdot y$. We have that the sum of profits is $rpMy$, and the sum of surplus value is $ew\ell y$. These sums are equal only if

$$(rpM - ew\ell)y = 0,$$

a relation which there is no reason a priori to assume. As a result, except on a set of zero measure of possible structures of production, we cannot have:

$$\frac{\text{sum of prices}}{\text{sum of values}} = \frac{\text{sum of profits}}{\text{sum of surplus value}}.$$

These two results, that the rate of profit is independent of the structure of production, and that value and surplus value are not conserved in the transformation, many Marxists prefer not to face. In my opinion, they are wrong in this, not only because Morishima and Samuelson offer a drop of comfort in the form of the "fundamental Marxian theorem," that the rate of profit is positive if and only if the rate of exploitation is also positive.[9] Marxists can face this situation because in fact it is possible within this conceptual framework to draw almost all the conclusions Marx tried to prove in his model.

1. *The Value of Commodities Recovered by Capitalists Is in Fact the Surplus Value*

In other words, even if the sum of profits is not the sum of surplus value, the value of the uses of profits is certainly the sum of surplus value. The proof of this theorem is quite trivial. It is enough to notice that in writing

$$p = (1 + r)\, pM,$$

we implicitly assume that the whole product is realized in sale, that there is no overproduction.

But the gross product $\mathscr{Y}$ produced in a period serves: to replace the inputs used up, $M\mathscr{Y}$; for the unproductive consumption of the capitalists, $\mathscr{C}$; and for the production of the elements of the expansion of production, or accumulation, $M\Delta\mathscr{Y}$. These two last terms constitute the use of the profit (the first term representing the uses of the initial capital). We have

$$\mathscr{Y} = M\mathscr{Y} + \mathscr{C} + M\Delta\mathscr{Y},$$
$$v(\mathscr{Y} - M\mathscr{Y}) = v(\mathscr{C} + M\Delta\mathscr{Y}),$$
$$ew\ell\mathscr{Y} = v(\mathscr{C} + M\Delta\mathscr{Y}),$$

which establishes the claim.

Intuitively we can see that if the prices at which capitalists sell the commodities differ from the values, the same thing happens with the commodities they buy, and the one difference balances out the other, in a sense which we must now make explicit.

2. *The Prices of Production Regulate the Behavior of Capitalist as "the Personification of Capital"*

In the last paragraph, the "unproductive consumption" of the capitalists was indeterminate, and it could not be otherwise at this level of abstraction. $\mathscr{C}$ could be determined only by considerations of a social–psychological

[9] See [13, 18]. We will present later a much stronger result.

nature. At this level, the capitalist is only, as Marx says, the "personification of his own capital," value which has only one goal, to expand. If then we reduce the capitalist to his essence, he would obey the famous protestant ethic of Max Weber, and the "duty to society" of Calvin: Frugal in the extreme, he would accumulate all his own capital, and invest it in his own sector, since there is no reason (at this level of abstraction) for him to do anything else.

Let then $p_i y^i(t)$ be the revenue of sector i in period t; it is completely used for the purchase of inputs to production in $y^i(t+1)$:

$$\begin{aligned} p_i y^i(t) &= (\mathfrak{p}M)_i\, y^i(t+1) \\ &= (1/(1+r))\, p_i y^i(t+1). \end{aligned}$$

Then

$$y^i(t+1) = (1+r)\, y^i(t).$$

On the other hand, these inputs to production (which include the subsistence goods for the newly recruited workers) represent the whole product of period t:

$$\mathscr{y}(t) = M\mathscr{y}(t+1)$$

so that:

$$\mathscr{y}(t+1) = (1+r)\, M\mathscr{y}(t+1).$$

The vector of production thus must be the righthand eigenvector $\mathscr{y}^*$ of the matrix M corresponding to the dominant eigenvalue, and grows exponentially at rate $1+r$. This is the famous growth path which Morishima calls the "Marx–von Neumann ray," and which I will call the "structure of integral accumulation," and which allows maximal balanced growth.

For this structure of production (which is of measure zero in the space of possible structures, but plays an undeniably special role) it is easy to show that, if we choose the covector of prices of production so that $\mathfrak{p}^* \cdot \mathscr{y}^* = \mathfrak{v} \cdot \mathscr{y}^*$, then we will have the sum of profits is equal to the sum of surplus value. In fact,

$$\text{sum of profits: } r\mathfrak{p}^* M\mathscr{y}^* = (r/(1+r))\, \mathfrak{p}^* \cdot \mathscr{y}^* = (r/(1+r))\, \mathfrak{v} \cdot \mathscr{y}^*,$$

$$\text{sum of surplus value: } \mathfrak{v} \cdot \mathscr{y}^* - \mathfrak{v}M\mathscr{y}^* = (r/(1+r))\, \mathfrak{v} \cdot \mathscr{y}^*.$$

3. *A Compensation Theorem for Values and Prices*

These two reference systems, $\mathfrak{p}^*$ and $\mathscr{y}^*$, will let us make rigorous the intuition, stated in paragraph 1, of a necessary compensation of the divergence between values and prices of production by the structure of production. Any product vector y can in fact be decomposed uniquely into a component y^* parallel to the integral structure of accumulation, and a component y' (which may have negative components) in the hyperplane orthogonal to the price system p^*:

$$y = y^* + y', \qquad \text{where} \quad p^* \cdot y^* = v \cdot y^* \text{ and } p^* \cdot y' = 0.$$

If we write

$$\delta v = p^* - v = \text{divergence of } p^* \text{ from } v,$$
$$\delta y = y - y^* = \text{divergence of } y \text{ from } y^*,$$

then

$$\delta v \cdot y + v \cdot \delta y = 0.$$

In fact,

$$\begin{aligned}\delta v \cdot y &= (p^* - v) \cdot y = p^* \cdot y^* + p^* \cdot y' - v \cdot y^* - v \cdot y' \\ &= -v \cdot y' = -v \cdot \delta y.\end{aligned}$$

To put it another way, if, starting from a path of reproduction corresponding to the structure of integral accumulation (and choosing the numeraire so that price is equal to value in total) we "deform" the structure of production in a certain direction, the measure of that deformation of production (in terms of values) is the measure of the difference between total price of production (and total value) for that structure of production. In this way we find a relation (certainly much weaker than Marx's claim) between the structure of production and the transformation of values into prices of production. But p^* and y^* remain functions of d, so that y is only a way of evaluating after the fact their relative variations.

4. *The Rate of Profit Is a Well-Defined Function of the Rate of Exploitation*

If one is not happy with the "fundamental Marxian theorem," one can try to express explicitly the relation between e and r (given the parameters A, d, and ℓ). But a function of the form $r = f(d, e)$ is subject to the objection that d and e are related, since $(1 + e) v \cdot d = 1$. We must separate more clearly the rate of exploitation from the bundle of goods workers consume. Two suggestions have been made as to how to do this.

The method of Dumenil and Roy [6] (see Appendix B) is to determine the workers' consumption bundle d in two steps, one dealing with the composition of the bundle, the other with the rate of exploitation. He writes:

$$d = wd^*,$$
$$v \cdot d^* = 1,$$

to express the idea that the workers choose a bundle of consumption goods from the simplex of bundles whose value is unity, and then scale this bundle to the rate of exploitation. The curves $r = f(d^*, e)$ are then defined by a simple relation; they are all convex, monotonically increasing in e, and bounded by R, the dominant eigenvalue of the matrix A of technical coefficients of production (which coincides with M when workers "live on thin air"). The envelope of this family of curves is bounded by the functions which correspond, for a given value of r, to consumption bundles d^* which maximise and minimise the organic composition of capital defined in a particular way.

Another, more complicated method has been proposed by Roemer [16] (see Appendix B). It allows workers to choose their consumption bundles individually. Each worker is assumed to have a preference ordering γ (with the usual neoclassical properties of convexity and continuity). Given the wage and the prices of production, each worker chooses freely how to spend his income. By using the Perron–Frobenius Theorem in conjunction with Brouwer's Fixed Point Theorem, Roemer shows that there exists a system of prices at which workers will choose bundles of goods which each have a value w. The curves are now indexed by the family of preference functions of the workers, Γ, but the shape of the family of curves $r = f(\Gamma, e)$ is the same.

Nevertheless, the use of Brouwer's theorem (which is an abstract result without direct economic relevance) raises a fundamental problem for this arguement. How could workers choose their bundle of consumption goods facing prices of production when their budget constraint is fixed in the system of values by the requirement $v \cdot d = w$? This problem is related to the very doubtful procedure which is common to all the Morishima-type solutions, the transformation of variable capital V on the basis of a given

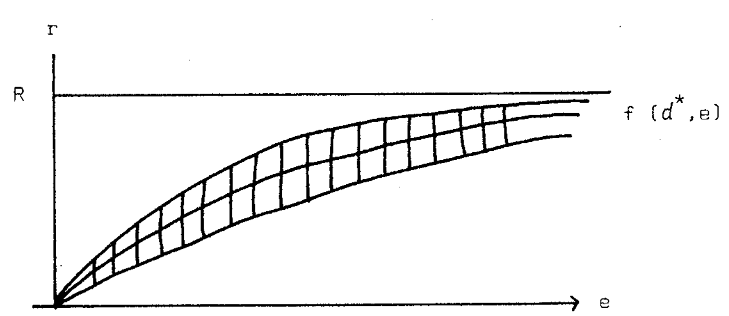

FIG. 1. Rate of exploitation and rate of profit.

volume of workers' consumption. To repeat the obvious, workers receive money, not a voucher for a bundle of commodities. Once the transformation has been achieved, and choices are made in the price system, the relation $r=f(d^*, e)$ is certainly true. But does it correspond to a causal relation between d^* and e on the one hand and r on the other? Certainly for Marx the causal relation was between y and e on the one hand and r on the other. Roemer's and Dumenil's contributions do not eliminate this weakness in the accepted solution to the transformation problem. We will return to this problem below.

5. *The Prices of Production and the Rate of Profit Are Logically Determined by the Labor Theory of Value and of Exploitation*

We have arrived at a critical point in the controversy. However strong may be the links between the system of values and surplus value on the one hand, and the system of prices and profit on the other, as we have just shown, these connections appear at first sight to be due to the fact that both systems depend on a third set of data, the given "technique" as expressed by the matrix $M = A + d \times \ell$.

In fact, the matrix M determines a "surplus," given only the technical condition that it is "viable," that is, produces more output than its inputs, and this net product is allocated according to two different principles: (a) in proportion to the labor expended during the last period on each commodity in the value system; (b) in proportion to the total costs of production (including both raw materials and labor) incurred in the last period.

Samuelson [18] compares these two schemes to the effects on the price system of a value-added tax on the one hand, and a turnover tax on the other. In the first case the solution of the system of equations determining prices is extremely simple, being a system of n simultaneous linear equations. In the second case we have to solve a polynomial equation of the nth degree (namely, $\det[I\mu - M] = 0$). Samuelson suggests:

> One might apply Marx's theory of the materialist determination of history to arrive at the hypothesis that it was Marx's incapacity in algebra and the absence of a

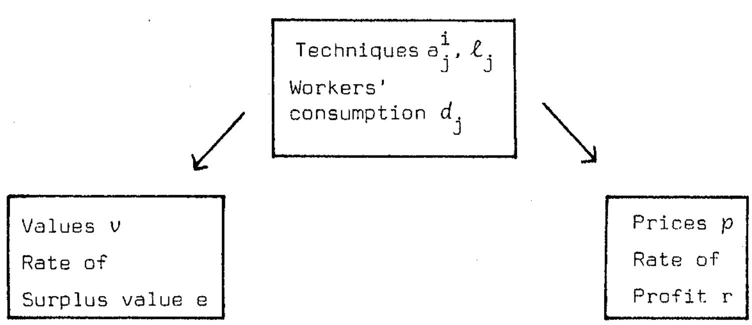

FIG. 2. The apparent common ancestry of values and prices.

computer that caused him to formulate his exploitation theory in Volume I terms which are unrealistic but which happen to be simpler to handle algebraically than Volume III's Walrasian relations. [18, p. 418]

The transformation problem can then be solved very easily, according to Samuelson (always on the basis of the given "technical" conditions A, d, and ℓ):

The "transformation algorithm" is precisely of the following form: "Contemplate two alternative and discordant systems. Write down one. Now transform by taking an eraser and rubbing it out. Then fill in the other one. Voila. You have completed your transformation algorithm." [18, p. 400]

Unfortunately, this pretty story assumes that the matrix M, from which are deduced, it is true, the price system p, and r, is given logically prior to the labor theory of value and exploitation. But we have already seen that this is not the case.

The idea that the "natural" or "technical" productiveness of the matrix M is the origin of surplus is an old idea, which Marx criticized when he found it in the post-Ricardians, especially J. S. Mill.

Favourable natural conditions alone give us only the possibility, never the reality, of surplus labour, nor, consequently, of surplus-value and of surplus product. These conditions affect surplus-labour only as natural limits, i.e., by fixing the points at which labour for others can begin. In proportion as industry advances, these natural limits recede. In the midst of our West European Society, where the labourer purchases the right to work for his own livelihood only by paying for it in surplus labour, the idea easily takes root that it is an inherent quantity of human labour to furnish a surplus-product. [12, Volume I, chapter XVI, p. 515].

Marx illustrates this point by describing a Pacific people who can meet their needs by working one day a week, until as a result of colonization they are forced to work the whole week.

How do the given techniques of production in the algebraic model conceal the social relations? Let us go over the logical chain step by step.

First we have the commodity character of the economy. From this we develop the substance and the form of value. For a given state of the productive forces (A, ℓ), we can derive the magnitudes v of the vector of values. Up to this point there is not a word about exploitation, nor about surplus value, nor about profit. Remember that ℓ expresses the quantities of abstract labor necessary to the production of the goods. These quantities might correspond to ℓ, 2ℓ, 3ℓ days of wage labor, depending on the intensity and duration of work.

We introduce capitalist exploitation, the sale and the use of the commodity "labor-power," characterised by the value of labor power and the duration λ and the intensity ε of work. The value of labor power, w, corresponds (through v, which we have defined already) to a bundle of goods, d, necessary to reproduce the labor power (at least this is how the Morishima-type argument proceeds.)

All these elements, λ, ε, w, d, and e, are directly or indirectly the objects of class struggle, and are related to each other in complicated ways.

λ and ε are weakly related to each other, and even more distantly connected to d. (We know that production falls only 1% when the length of the working day falls by 2%, for example). We can therefore consider λ and ε logically prior to e and to the amount of surplus value. They are implied in the definition of the linear transformation which maps covectors of the quantity "labor power" into covectors of "value added." This mapping, which is an identity when the units are properly chosen, is necessary to link the "technical" data to the equations defining prices of production:

$$p = (1 + r)(pA + w\ell) \qquad \text{(where } w = \text{wage rate)}$$

being only an abbreviation for:

$$p = (1 + r)(pA + w\ell T^{-1}).$$

λ and ε being given, however, the relation between e, d, and w is much more complicated. The dynamic process that appears at first glance to be playing the decisive role is from e to w to d. That is, capitalists and workers confront each other over the division of the value added, the result defining the value of labor power w, which workers spend in money as they wish. Hence, this defines, through the system of values or prices, the bundle of commodities d which the workers can eventually purchase.

Nevertheless, Marx, in Chapter VI of *Capital*, anxious to give an intuitive sense to the formulation that "labor power, like any commodity, has a value, namely, the quantity of labor necessary to its reproduction" stretches the analogy to some degree in supposing the existence of a bundle of workers' consumption, a kind of vector input to the household-enterprise (whose worker, let us remark in passing, the wife, works for free, and whose proprietor sells the product at its cost). Leontieff, von Neumann, and Morishima codify this reduction of the worker to a beast of burden which needs its feed.

Although Marx quickly abandons this physical determination of a bundle of wage goods, and studies the value of labor power as a "quantity of paid labor,"[10] his assumption that there exists at any given moment a "standard of worker's consumption" is not indefensible. Certainly unions do not bargain directly for washing machines or color televion sets; to repeat yet

[10] Samuelson quite freely admits this [18, p. 422], so that it is surprising that he accepts Seton's solution (which depends on the concept of a physical bundle) as a solution to Marx's transformation problem.

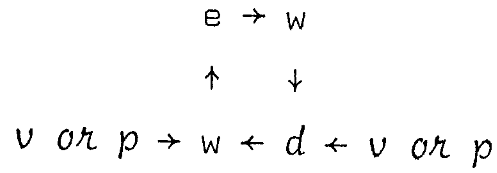

FIG. 3. The "loop" $e - d$.

again, they bargain over increases in wages. Nevertheless, a "standard of living" once it is widely adopted, can be lowered only with difficulty, not for moral reasons, but because of the vested interest of the sectors which produce those consumption goods.[11] So there is a feed-back from the historical standard of living (given by the bundle d) to the value w (and hence to e) through the system of prices or values.

We could say that e and d are "dialectically" related, d being the base, and e the directing factor. The logical chain looks like Fig. 3.

Once we have fixed d, and required that wage cover the cost of d at the prices of production, we can move to the last link in the logical chain, summed up in Fig. 4. The striking comparison with Fig. 2 justifies the careful way we have uncovered the meaning of the algebraic symbolism used in the accepted transformation. Even if we take d as given, the labor theory of value and of exploitation appear clearly as logically prior to the analysis of prices of production.

III. A New Solution to the Transformation Problem

The usually accepted solution to the transformation problem, then, has now been shown to be less inconsistent with Marx's theses than many scholars think. Nevertheless, it still deviates from Marx's ideas in significant ways.

(a) If the labor theory of value and of exploitation is logically prerequisite to the calculation of prices of production, this fact is not clearly revealed in the usual algorithm of transformation. One would like to "see" the value being redistributed over the commodities in the process of equalization of rates of profit.

(b) If there exists a relation between the structure of output and the deviation of prices from values, it appears "ex-post" in the usual treatments of the problem. But in Marx's treatment the structure of output (which determines, through the weights applied to the different organic compositions

[11] This is a critical point in understanding the inflationary form of the current crisis. See [1, 2, 9].

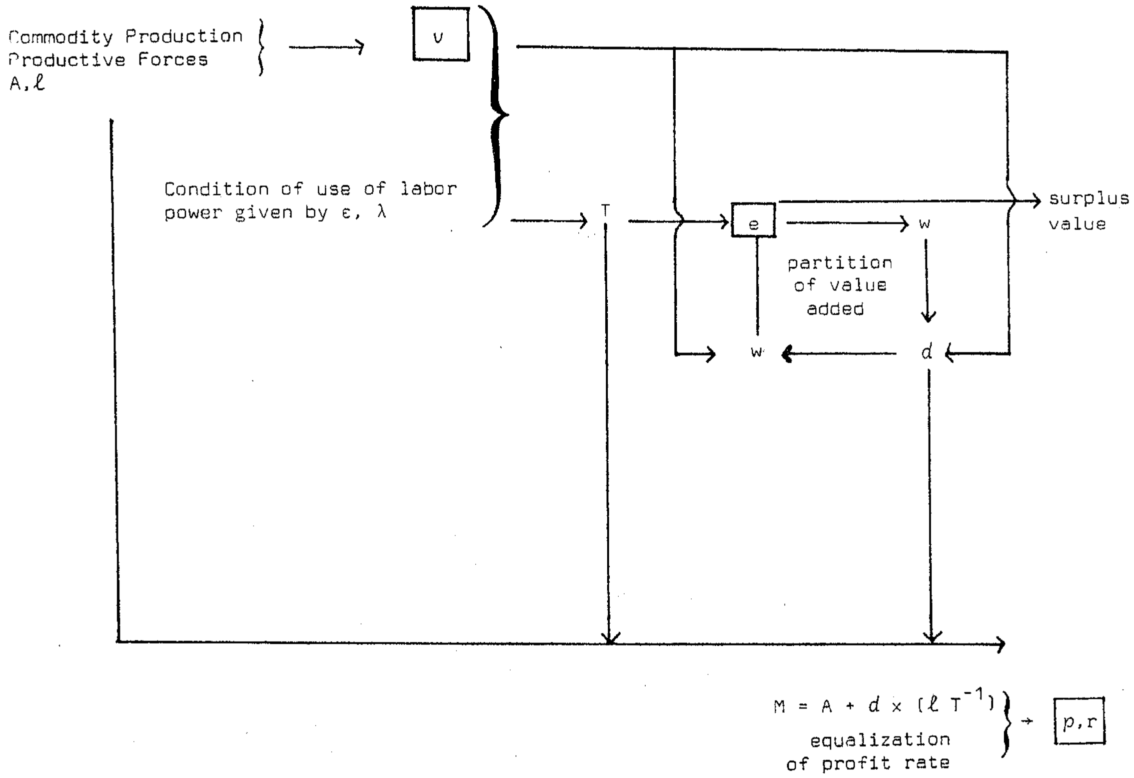

FIG. 4. The conceptual filiation from value and exploitation to prices and profits.

of capital, the quantity of surplus value to be distributed among the invested capitals) appears as a determining parameter for the rate of profit and prices of production. In the usual treatment the structure of workers' consumption seems to play this determining role.

1. *The Root of the Question*[12]

We have seen that all the anomalies between Marx's intuitions and the Morishima-type solutions have their origin in the way Bortkiewicz and those who follow in his tradition have treated the problem of the transformation of the value of labor power. They all treat V (or in the present notation, w) exactly like constant capital, C. No one would deny that if the technique of production, (a_j^i, l_j) is given[13] the cost price of constant capital C is $\Sigma p_i a_j^i$.

[12] The credit for the discovery of this new solution belongs chiefly to Dumenil [7], who identified the two points necessary to carry it out: the interpretation of the value of labor power, w, as a share of value added; and the application of the transformation to the net product. His arguments rest on a deep understanding of the conceptual framework of *Capital*. The responsibility for the present exposition, which follows my own analysis in [9], and for the arguments in the next paragraphs, is mine.

[13] Of course, the least costly technique could depend on e and on r. Even worse, a technique could be the cheapest in value, but not the cheapest in prices of production. Here we will ignore these dynamic problems (see [10]).

But can we reduce labor power to an output requiring as input a vector d with a cost price $\Sigma p_i d^i$? Such a formulation would amount in fact to a model of a "quasi-slave" mode of production, that is, a mode of production where slavery exists within the units of production, but relations between units of production are commodity relations (as was the case in the Southern states of the U.S. before the Civil War).[14] But in fact workers in developed capitalist society receive a money wage which they can choose "voluntarily" to spend according to their needs. Certainly the general standard of living limits their choice to a certain fuzzy set of bundles, and the value of bundles in that fuzzy set at a given time in the history of class struggle is a basis for the value of labor power. But this does not justify our adopting a scheme of unilateral causality like the one shown in Fig. 2. It is much more accurate to think of matters as in Fig. 3, with a "loop" between d and e.

But here a problem arises. The budget constraint of workers in reality is expressed in the same system of prices as the commodities they buy. But in Morishima-type solutions the budget constraint is expressed in values, while the system of prices depends of the bundle of commodities finally chosen. This difficulty is resolved only in form by Roemer's use of the fixed point theorem [16].

On the other hand, matters are much simplified if we interpret w as a "quantity of paid labor," a share $1/(1+e)$ of the value added, the money equivalent of which is to be spent at prices of production to meet socially determined needs. The prices of production then should not depend on d, but, as in Marx, on y. The result is a bundle of consumption goods d whose value in terms of labor embodied (though not in terms of money, that is abstract social labor) could well be different from w. This pattern of consumption d, if it does not satisfy the workers, could then become the basis for a renegotiation of the proportion e of the value added received by the workers.

This solution, which requires us to treat constant and variable capital differently, turns out to be consistent with the ideas Marx left concerning the transformation problem. This is obviously not an argument for its correctness, but insofar as students of the transformation problem have tried to formalize Marx's model in order to show that it is inconsistent, it is important to make sure that they have in fact formalized Marx's model, and not somebody else's. What does Màrx tell us, after he has himself criticized his clumsy little model of the transformation? That it is necessary to transform the elements of cost prices as follows:

[14] In any case, labor power could not be identified (as in Samuelson) with the product of sector 0, since, in the enterprises of that sector (the households) labor (the wife's) would be unpaid and the "boss" would sell the product without receiving any profit. It might be otherwise if workers lived in boarding houses run by capitalists, but then the price of labor power would be $(1+r)p \cdot d$, not $p \cdot d$.

> So far as the constant portion is concerned, it is itself equal to the cost-price plus the surplus-value, here therefore equal to cost-price plus profit, and this profit may again be greater or smaller than the surplus-value for which it stands. As for the variable capital, the average daily wage is indeed always equal to the value produced in the number of hours the labourer must work to produce the necessities of life. But this number of hours is in his turn obscured by the deviation of the prices of production of the necessities of life from their value. [12, Vol. III, p. 161]

In other words Marx thought that in the case of constant capital it was necessary to transform the value of commodities. But in the case of variable capital, the wage, insofar as it represents a share of the value added, that is, as a "number of hours," is conserved by the transformation, while the labor time itself, considered as the equivalent of a particular bundle of commodities, is transformed. This is exactly what happens in the solution just proposed.

It remains to check up that this solution is mathematically consistent.

2. *The Model*

Suppose v is the vector of the values of the commodities, determined by A and ℓ. Suppose we have a given rate of exploitation e and a value of labor power w with $w = 1/(1+e)$. We seek a redistribution of the total value added (that is, of the global flow of abstract labor) produced in the period, over the *net* product y of that same period.

Here we meet a second criticism of the currently accepted solutions to the transformation problem. For the most part, this work tries to verify the "Marxian equalities" of the type "sum of prices equals sum of values," and so on, without posing the question: "sum of which prices?" Marx observed that we cannot aggregate all prices or all profits and hope to reach the sum of values or of surplus value:

> In applying this approach to the aggregate product of society, we must make some rectifications. Looking upon society as a whole, the profit contained in, say, the price of flax cannot appear twice—not both as a portion of the linen price and as the profit of the flax. [12, Vol. III, p. 160]

Today, thanks in part to the work of Marx in Volume II of *Capital*, written after Volume III, on reproduction, this "rectification" is easy to make, since we understand the relation between gross and net product.

Thus the redistribution of value added is indeed a redistribution, in that it reallocates a constant quantity of the substance abstract labor. Let us call p_i^* the value reallocated to each commodity i. The vector p^* of reallocated values defines the system of relative prices of production (the level of prices depending on the choice of the numeraire). We must have, by definition

$$p^* \cdot y = v \cdot y. \qquad \text{(H1)}$$

On the other hand, this reallocation ought to achieve an equalization of profit rates on invested capital, so that the value redistributed to good i ought to be equal to $(1+r)$ times the sum of constant capital (evaluated in redistributed values) and of variable capital (measured by the value paid to workers in exchange for their giving over their labor power), $(1+r)$ being the same in all sectors. Then, if we assume as we have all along that the choice of units of quantities of labor power and of abstract labor expended are such that the linear transformation T may remain implicit, we must have (as a definition, the consistency of which with the earlier condition must be checked):

$$\not{p}^* = (1+r)(\not{p}A + w\ell). \tag{H2}$$

We can define this redistribution in another way. We want the "labor commanded"[15] directly (during the last period, period 0), and indirectly (in earlier periods 1, 2,...) to produce good j[16] to contribute to the redistributed value $\not{p}^*$ of the product in the proportions $(1+r)^{n+1}$.

$$\not{p} = (1+r)w\sum_{0}^{\infty}((1+r)^n\,\ell A^n). \tag{H2$'$}$$

Equation (H2′) is equivalent to (H2), which can be written as

$$\not{p} = w\ell(I/(1+r) - A)^{-1}, \tag{H2$''$}$$

which gives (H2′) when we expand the inverse matrix as a series.[17]

[15] "Labor commanded" is a bastard concept of Classical Political Economy (Smith and Ricardo), which Marx uses explicitly in *Theories of Surplus Value* and implicitly in Volume III of *Capital*. There it designates the value or price of purchased manpower when used as an index of labor expended. In other words, it is variable capital V in so far as it can serve as an index of $V+S$, that is, in so far as we can hold w and the transformation T constant. For instance, when Marx denotes the organic composition of capital as C/V in Volume III he remarks explicitly that V is used as an index of embodied labor. Hence the difficulty many writers have had who do not see why the rise in the organic composition of capital eventually involves a fall in the rate of profit. If we understand (as Marx did) that a rise in the organic composition of capital is in fact an increase in $C/(V+S)$, we can easily see that the rate of profit must tend uniformly to zero with the growth of the organic composition of capital, whatever may happen to $e=S/V$. (On "labor commanded" and its traps, see [4, 9].)

[16] The quantity of abstract labor embodied directly in j is ℓ_j; the quantity directly embodied in its means of production is $\ell \cdot A_j$; in the means of production of the means of production $\ell \cdot [A^2]_j$; and so forth. Since T is the unity tensor, one gets "labor commanded" by multiplying by w.

[17] The series expressed in (H2) will converge to the inverse matrix as long as $(1+r) < 1+R$, where $1/(1+R)$ is the dominant eigenvalue of the matrix A. This condition will hold, as we shall see, as long as workers consume something, since the matrix A is productive. (See [15]).

Notice that the same operation can be applied to

$$v = (1 + e)\, w\ell(I - A)^{-1},$$

which yields

$$v = (1 + e) w \left(\sum_0^{\infty} \ell A^n \right).$$

Now we can see that redistribution consists in reallocating the whole value[18] in such a way that the surplus value is proportional, not to the simple sum of the labor commanded in earlier periods, but to the sum of labor weighted by a factor $(1 + r)^{n+1}$. Samuelson's analogy to the case of a value added tax and a turnover tax is correct, but not by itself enough, since the "turnover" concept is not well defined. Suppose a weaving firm buys a spinning firm. The labor of the spinners, which before appeared as constant capital in the weaving sector (according to (H2)) now appears as variable capital, although it is tied up for two periods instead of one. Equation (H2′) correctly handles this situation.

The system of redistributed value which we seek (and from which we can derive the whole system of prices of production once the numeraire is given) is now well defined by (H1) and (H2). The questions remain whether such a system exists, and if it does exist, what properties it has. We can prove the following theorem, which sums up Marx's conclusions concerning the transformation problem, quite easily.

THE MARXIST TRANSFORMATION THEOREM. 1. *For every structure of output, there exists one and only one capitalist redistribution of value which equalizes rates of profit.*

2. *If we choose the numeraire so that the sum of value added in value terms is equal to the sum of prices of net product, then the sum of profits is equal to the sum of surplus value.*

3. *The average rate of profit is a function of the rate of exploitation, of the technical coefficients of production in each sector, and of the allocation of social labor among the sectors, and thus of the structure of output.*

Proof. 1. For all y, there exists $(p^*, (1 + r))$ with

$$p^* \cdot y = v \cdot y, \tag{H1}$$

$$p^* = (1 + r)(pA + w\ell). \tag{H2}$$

[18] This decomposition of the "constant capital," into variable capital and surplus value (or, transformed, of the price of intermediate goods into wages and profit) was already known to Adam Smith.

In fact, when we write (H2) as (H2′), we see that p^* is a continuous increasing function of $(1+r)$, so that $p^* \cdot y$ is a continuous increasing function of $(1+r)$ as well.

When $r=0$, $p^* \cdot y = v \cdot y/(1+e) < v \cdot y$.

As r approaches the maximum profit rate in (H2′), $p^* \cdot y$ tends to infinity (see footnote 17). Thus there exists one and only one value of r and one and only one vector p^*, which satisfies (H1). The critical rate r must be positive[19] and depends on y.

2. Let y be the net product (in the sense of national income accounting) which embodies the value added, and $\mathscr{Y}$ the corresponding vector of gross outputs. By definition:

$$p^* \cdot y = v \cdot y \qquad \text{and} \qquad \mathscr{Y}(I-A) = y.$$

We have that

$$\begin{aligned} \text{the sum of profits} &= p^* \cdot y - w\ell \cdot \mathscr{Y} \\ &= v(I-A)\mathscr{Y} - w\ell \cdot \mathscr{Y} \\ &= ew\ell \cdot \mathscr{Y}, \end{aligned}$$

which is the sum of the surplus values.

3. $r =$ sum of profits/invested capital

$$= ew\ell \cdot \mathscr{Y}/(p^*A + w\ell) \cdot \mathscr{Y}.$$

Thus

$$r = e/((p^*A\mathscr{Y}/w\ell\mathscr{Y}) + 1)$$

and

$$p^*A\mathscr{Y}/w\ell \cdot \mathscr{Y} = \sum (p^* \cdot A_j/wl_j)(l_jY_j/\ell \cdot \mathscr{Y}).$$

This is the average of the organic compositions of capital in the various sectors, each weighted by the share of wage labor employed in that sector, evaluated at prices of production. These organic compositions are functions of the technical composition of capital in the sectors, and of the system of

[19] We can see that for r to be positive, it is necessary and sufficient that w be less than 1, or that e be positive. This establishes in this model the "Fundamental Marxian Theorem" of Morishima and Okishio.

prices of production which itself depends only on e, $\mathscr{y}$, and the technical coefficients a_j^i and l_j.[20]

This last formula has the defect that it presupposes a knowledge of $\mathscr{p}^*$. I have written it because of its simularity to Marx's formula, which Marx himself recognized to be only a first approximation.[21]

We can compute r as a direct function of the basic parameters, using the method Dumenil and Roy employed to study the Morishima-type solution.

We start with $\mathscr{p}^* \cdot \mathscr{y} = \mathscr{v} \cdot \mathscr{y}$.

Using (H2″) and the fact that $w(1+e)=1$, we get

$$w\ell(I/(1+r)-A)^{-1}\mathscr{y} = (1+e)\, w\mathscr{v} \cdot \mathscr{y}$$

and

$$e = \ell(I/(1+r)-A)^{-1}\mathscr{y}/\mathscr{v} \cdot \mathscr{y} - 1.$$

This rate depends only on the structure of output $\mathscr{y}$. Let $\mathscr{y}^*$ be parallel to $\mathscr{y}$ with $\mathscr{v} \cdot \mathscr{y}^* = 1$, and we have

$$e = \ell(I/(1+r)-A)^{-1}\mathscr{y}^* - \mathscr{v} \cdot \mathscr{y}^*$$

Since

$$\ell = \mathscr{v}(I-A),$$

we get

$$\begin{aligned} e &= \mathscr{v}((I-A)(I/(1+r)-A)^{-1} - I)\,\mathscr{y}^* \\ &= \mathscr{v}((rI/(1+r) + I/(1+r) - A)(I/(1+r)-A)^{-1} - I)\,\mathscr{y}^*, \\ e &= r\mathscr{v}(I-(1+r)A)^{-1}\,\mathscr{y}^*. \end{aligned}$$

e is thus a continuous increasing function of r, so that we can invert it to get $r = f(\mathscr{y}^*, e)$.

This function takes the place of the relation $r{:}\, f(\mathscr{d}^*, e)$ of the last section. The structure of output $\mathscr{y}^*$ takes the place of the structure of workers' consumption $\mathscr{d}^*$. The curve relating r to e is increasing, concave, has an

[20] Of course these "technical" coefficients a_j^i, l_j are the material expression of social relations: the specialization of labor, taylorism, fordism, and so on. By "technical" I mean only that these coefficients are given before the conditions of appropriation of surplus value, before T, e, w are given, and therefore (and here is an essential difference from the Morishima-type of solution) completely independently of $\mathscr{d}$.

[21] In Marx's approximation, which is $r = \Sigma S_i/\Sigma(C_i + V_i)$, the organic compositions of capital are evaluated in the labor values. This is the *only* consequence of Marx's use of this approximation.

asymptote at $r=R$, where $1/(1+R)$ is the dominant eigenvalue of A, and a finite slope at the origin.[22] The envelope of the family indexed by y^* is composed, as in Section II.4, of segments of particular curves (the ones corresponding to the y^* which maximize and minimize the organic composition of capital at a given r).

IV. Comparison of the Two Solutions

We have now reached, in the new solution (which I will call "system B" from now on), nearly all the conclusions Marx expected from the transformation of values into prices of production. Are the conclusions drawn from Morishima-type solutions (from now on "system A") false? Of course not: They are mathematically correct, and economically compatible with the marxist theory of value and of exploitation. Suppose we have made the transformation for given e and w, according to method B. We now know p and r. For this system of prices, this wage, and this rate of profit, suppose that workers choose a bundle of goods d. Then using bundle d in method A, we must arrive at the same p and r. Nevertheless, we know that for given e or w, the two solutions A and B do not give the same results: One violates the marxian equality of profits and surplus value, the other does not.

There is no contradiction here: The simple point is that e and w do not have the same meaning, nor the same quantitative measure, in the two systems, though they serve as indices to represent the same theoretical concepts.

In system A, $w(A)$ is the value of what workers consume:

$$w(A) = v \cdot d.$$

In system B, $w(B)$ is the part of the value, which workers have created, to which they have received a claim in the form of wages to spend on the market where prices are regulated by the redistributed value.

$$w(B) = 1/(1+e(B)).$$

There is no reason to expect that $w(A)=w(B)$, and that $e(A)=e(B)$. In fact, in general, the value embodied in the uses of wages is not equal to the part of the produced value paid to the workers and spent by them according to the system of redistributed value.

$$w(A) = v \cdot d \neq w(B) = 1/(1+e(B)).$$

[22] If we work out the series expression for $e=f(y^*, r)$, it is obvious that, for all r, d^2e/dr^2 and de/dr are positive and that, at the origin, e/r tends to $v(I-A)^{-1}y^*$.

Once the transformation has taken place, for a given y, according to system B, does there exist a structure of workers' consumption d^* such that $w(B) = v \cdot d^*$? The answer is clear from the analysis of Section II-2 of this paper. The required d must be chosen so that y is the right eigenvalue of the matrix $M = A + d \times \ell$. (We shall not discuss here in what interval this inverse mapping is one-to-one and continuous). In every other case, all the results hold good separately, but $e(A) \neq e(B)$ and $e(A) = f(y^*, d^*, e(B))$, where $f(y^*, d^*, \cdot)$ is the composition of the inverse of $f(d^*, \cdot)$ and $f(y^*, \cdot)$. These results are summed up in Table I.

Solution B is not only closer to the intuitions in Marx, but it is also much easier to manipulate mathematically. Must we then relegate the "old" solution, the fruit of a half-century's work, to the museum of curiosities in

TABLE I

Comparison of the Two Solutions

System A	System B
v = vector of labor values = labor time embodied in commodities	
e = division of embodied labor time between paid and unpaid labor.	
$e(A)$ is defined as the value embodied in workers' consumption	$e(B)$ is given a priori and workers use the wage to buy commodities at their transformed prices
vd is given a priori	vd is determined after prices of production
r and p are unique (up to scalar multiplication) when:	
d is given	y is given
$r = f(d^*, e(A))$	$r = f(y^*, e(B))$
r varies with the structure of workers' consumption	r varies with the structure of net output
r does not vary with the structure of net output	r does not vary with the structure of workers' consumption
If the numeraire is chosen so that the sum of prices = the sum of values	
Σ profits $\neq \Sigma$ surplus value unless $y = y^*(d)$	Σ wages $\neq \Sigma$ values consumed unless $d = d^*(y)$
but Σ values of the uses of profit $= \Sigma$ surplus value	but Σ profits $= \Sigma$ surplus value

the history of economic thought? I do not think so, because this solution has forced us to explore carefully (as I tried to do in Part II of this paper) the conceptual context of the transformation problem; that is, all those problems connected to the "realisability" of the pair (y, d). In fact, the new solution, precisely because of its simplicity, does not use at all the assumption that the net production y is realised in some balanced model of accumulation. What we gain by expressing r directly as a function of e, and y^*, we lose through the complete indeterminacy of y^*.

Let us take an example (another one is sketched in Appendix C).

Suppose $w(B)$ is the part of the value added which is paid to workers, which they hasten to spend on necessities, and if possible on discretionary items. But, surprisingly, if the structure of production changes, so that the price system changes, as a consequence the frontier of the workers' budget set will move. At a single rate of exploitation, and value of labor power, workers might be able to afford both necessities and some luxuries, or not even their necessities, depending upon the structure of production. This does not fit very well with marxist intuition.

In system A the same phenomenon appears in the following way. Once the value of labor power, $w(A)$ is given, the average rate of profit depends on the structure of workers' consumption, d^*. I do not find this very bothersome.[23] This is simply the guise, transformed and enriched, which the theory of relative surplus value in Volume I of *Capital* takes in the transformation problem. Marx means by relative surplus value the increase in the rate of surplus value which arises from a fall in the value of the goods in the bundle d. But let us suppose a technological change which leaves invariant the value of d, but reduces the technical composition of capital in the sectors producing the elements of d. There is no relative surplus value generated by this change, but the rate of profit, intuitively, ought to rise. This conclusion is made explicit in solution A, which argues that given the value of labor power, the rate of profit will depend on the distribution of workers' consumption among sectors with different compositions of capital.[24]

In contemporary capitalism, characterised by a tight connection—because of the necessity of realising productivity gains—between the substitution of machines for people in production and the spread of higher standards of living to workers (what Gramsci calls "fordism"), the vectors y and d are

[23] This is, of course, a subjective opinion. Dumenil found this conclusion so unacceptable that it moved him to develop solution B.

[24] Here I am mixing up comparative static and dynamic arguments. But we could just as well imagine two goods which are perfect substitutes and have the same value, but are produced in sectors with different technical compositions of capital. If a large number of workers choose one rather than the other, the effect would be the same as that of a technical change of the type I have described.

closely linked by complex dynamic processes[25] which are the foundation of the dialectical play, the loop of Fig. 3. Under these circumstances, it is helpful to be able to call on both solutions A and B.

But we now are raising questions concerning the development of capitalism, which involve the extremely difficult problems of the contradictory pressures on the rate of profit, of crises of realisation, of inflation, and so on. These basic problems Marx and his successors have looked on as having the highest priority, leaving to one side the technical exercise which Samuelson may not have been wrong, in the end, to call the "so-called" transformation problem.[26]

APPENDIX A: The "Tensor of Exploitation"

Marx has analyzed the confusion (common in Smith and Ricardo) between the "value of labor power" and the "value added by labor," and between the "value of a commodity measured by the labor incorporated in it" and the "value of a commodity measured by the quantity of labor (that is, labor power) which it can buy, or command," the two confusions being closely connected. (See K. Marx, *Theories on Surplus Value*, Chapter III.)

It is clear that if we know the duration and the intensity of labor, we can calculate the quantity of labor expended in a day by a unit of purchased labor power m.

Let $\mathscr{m}$ be the n-tuple m_j of quantities of labor power which must be purchased in order to produce one unit of commodity j. This n-tuple is a linear form on the space of commodities, as is the covector ℓ: But instead of measuring the quantities of social labor expended, it measures the quantities of labor power hired by the capitalists. This is not the same linear form as ℓ: They measure different things, and the components of the vectors will also differ (unless we make a special choice of units). But we can transform one into the other by a linear transformation, just as in the study of elastic bodies we can find the vector of strains from the vector of displacements. The mathematical theory of such relations is the theory of tensors, linear operators on vectors and covectors. Here we need a very simple tensor, a "1-covariant, 1-contravariant" tensor T, which can be represented as a matrix:

$$\ell = \mathscr{m}T.$$

[25] These mechanisms are essentially what I call "the monopoly regime of intensive accumulation" (see [9]).

[26] Since the completion of this paper, the new solution has been shown to be easily generalized to production systems which involve fixed capital and rents. This confirms the result that Marx saw as central, that when the numeraire is chosen so that the total price of the net product is equal to the value added, the sum of net profits and rents (and any other revenues of unproductive classes) is equal to the sum of surplus value. See Lipietz [11].

This matrix is the identity matrix I if we choose units appropriately, as I explicitly suggest in the text, and as is assumed implicitly by most students of the transformation problem. In the general case:

$$T_i^i = \varepsilon_i \lambda_i,$$
$$T_j^i = 0 \qquad \text{if} \quad i \neq j.$$

This formulation has the advantage of making explicit the two fundamental givens of capitalist exploitation, the duration and the intensity of labor which may eventually be allowed to differ between different sectors (as indeed the wage may vary as well). In the last case, the scalar e ought itself to be replaced by a tensor.

The differentiation of rates of exploitation between sectors which we can express using the tensor representation still would be of limited scientific interest. As Marx recognizes himself, there exist differences in the rate of exploitation, and Engels notes that these differences are likely to be greater than the differences in the rate of profit, since the "equalizing forces are stronger here than there." [9, p. 268]. But two arguments intervene here:

(a) First, taking account of differences in the rate of exploitation misses the point. The crux of the "transformation problem" is that, within the pure labor theory of value, the rate of profit and the rate of surplus value cannot both be equalized across sectors unless the composition of capital is also equal across sectors. But the equality of these two rates is implied by the level of abstraction at which the argument takes place: all the members of one class are equal in their confrontation with the other class (so that the rate of exploitation must be uniform) and in their confrontation with members of their own class (so that wages and profit rates must also be uniform).

(b) If we want a better conceptualisation of concrete reality, we may have to take into account differences between individual members of the two classes. But why privilege sectoral differences in this treatment? The conditions of exploitation in fact probably differ much more by sex, race, region, than by sector.

APPENDIX B: THE MAPPING THAT RELATES THE RATE OF PROFIT TO THE RATE OF EXPLOITATION

(a) *Dumenil's Solution* [6]

Dumenil and Roy compute the inverse function: $e = rv(I - (1+r)A)^{-1} d^*$.

Since d^* is a convex combination of bundles containing only one good, e

lies, for a given value of r, between the extreme points of curves corresponding to these basis bundles. If we trace all these curves (which may intersect) we will have the envelope of the $f(d^*, \cdot)$ functions.

Dumenil also puts forward a more classical expression for f. If y' is the vector of activities which produces net output d (that is, $y' - Ay' = d$), we have $r = e/(1 + (pAy')/(p \cdot d\ell \cdot y'))$, which reproduces Marx's classic formula for the rate of profit, $r = e/(1+q)$, where q is the average organic composition of capital. Here the organic compositions are evaluated at prices of production, and aggregated with the weights given by the vector y'. This result, which is not obvious, allows us to study specifically which curves make up the envelope for a given r.

(b) *Roemer's Solution* [16]

Here is a sketch of Roemer's proof. To a total consumption vector $\mathscr{D}$ satisfying the constraint on w, there corresponds, by the Perron–Frobenius theorem, a system of prices of production p (the wage being taken as the numeraire). To this system p corresponds (through the preference orderings) a total consumption $\mathscr{D}'$. This is reduced by scalar multiplication to a $\mathscr{D}''$ which satisfies the constraint on w. The mapping from $\mathscr{D}$ to $\mathscr{D}''$ is continuous on a compact convex set, so that Brouwer's theorem guarantees the existence of $\mathscr{D} = \mathscr{D}''$. Roemer shows that all the fixed points of this mapping correspond to the same system of prices and the same rate of profit.

APPENDIX C: THE PROBLEM OF NON-BASIC GOODS

Here is another example concerning the indeterminacy of net product in the solutions of type B, which I will not develop here because it relaxes the assumption that the matrix M is indecomposable, and a full discussion would make this paper too long. We know that if there exist non-basic sectors (luxury goods, for example), the rate of profit is determined, according to method A, only by the subsystem of basic commodities. This result bothers some marxists: Why should the average rate of profit not be influenced by the surplus value appropriated in the production of luxury goods in "Department III"? The intuition underlying this question is that if Department III has a very low organic composition of capital, it should be possible to raise the average rate of profit by employing a larger fraction of the labor force in that Department. This intuition depends on the apparent indeterminacy of y in solution B. But this appearance is misleading. The production of Department III must be realised, and leaving aside the consumption of capitalist households which receive the profits of Department III—the output of Department III must be purchased out of the surplus value generated in other Departments. For example, in simple reproduction, the value of the capital enganged in Section III must equal the surplus-value produced in the two fundamental sections (straightforward proof). It turns out that when we take account of this connection, we can reach the same conclusions as in method A. But this gives us directly the conclusion that, once the consumption of workers, d, is given, whatever may be the structure of net output y, given only that it allows for full realization, the rate of profit remains the same. This means that, once workers' consumption is fixed, the net product, whatever assumption we make about the regime of accumulation, is constrained to a region in which profit rate is an invariant determined by solution A, this region being the inverse image of $r=f(d^*, e_A)$ for the function $r=f(\cdot, e_B)$ with $e_B=f(\cdot, d^*, e_A)$. This elegant result ought not to be overlooked.

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The So-Called "Transformation Problem" Revisited: A Brief Reply to Brief Comments

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If I interpret correctly the comments on my text by Duménil and Flaschel, my answer has to be very brief, since they basically agree with my paper.

I. Answer to G. Dumenil

His comment is two-fold.

1. *The Developments on the "Old" Solution are Useless*

According to Duménil, the "new" solution, being the "right" interpretation of Marx's thought, supersedes the "old" one as "contradicting Marx's aim in Capital."

In substance, the difference between the two solutions lies in the definition of the value of labor-power: either as the value of workers' consumption d, or as the wage share w of value added. Duménil thinks that one cannot defend any interpretation through considering the way this value is determined, but only on the theoretical basis of its fidelity to Marx's text. That is a defendable point of view in marxology. But for "marxist economics," the problem is of the relevance of the theory. Formally, either of the two definitions could be accepted.

My position is the following:

(1) If one chooses the first position, one must adopt a definition of surplus-value consistent with the definition of the value of labor-power. If the value of labor-power is the value of the uses of wages, the amount of surplus-value is the value of the uses of profits. Thus, the "marxian equality:" "sum of surplus-value = sum of profits" holds. (II-1 in my paper).

On this point, the "standard transformationists" are inconsistent. On the one hand, they define the value of labor-power as the value of the uses of

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wages. On the other hand, they are astonished that the amount of profit (measured in money) is not the amount of surplus-value (measured in labor), the labor-value of money being defined as an average at the level of aggregate net (or gross) product through the equality "sum of prices = sum of values." But this equality could not hold, since the "average value-price divergence" has no reason to be the same at the *aggregate net product* level and at the *aggregate capitalists' demand* level. The equality of average value of money at these two levels (and thus the equality of sum of profits and sum of surplus-value) is secured in the case of "integral accumulation" (point II-2), and otherwise the divergence is explained in point II-3 of my text.

In the third part of my paper, I use the second definition. Then the value of labor force is the money wage multiplied by the mean value of money at the aggregate net product level, which is chosen (through Eq. H_1) as equal to 1 (thus, the value of the labor-power is expressed by the same number as the unit wage). And I prove quite easily that the amount of surplus-value expressed in money is conserved in spite of its reallocation into profits.

It should be noted that Foley [5] reached very similar conclusions, in 1979, independently from Duménil's and my findings, through an investigation of the questions "what is the value of money? of a wage expressed in money?."

(2) Both definitions could be defined through references to Marx, though Marx clearly thinks of the second one in Volume III of "Capital." But one must keep in mind the first one as far as one studies such problems as growth rate, reproduction scheme, and so on (cf. Lipietz [6] or Duménil [4]!).

(3) There is an historical dimension of this problem. In the 19th century, the class struggle within distribution rested on the defense of *d*. In the "fordist" regime of accumulation (see [1, 7]), the shrinking of the unit value of commodities proceeded so rapidly that the growth of the purchasing power of wage became the center of social conflict (in French official terms: "le partage des fruits de la croissance entre partenaires sociaux"!). Thus, *e* (the partition of value added) became the leading factor of the determination of distribution.

2. *The Mathematical Demonstration of the "New" Solution is Useless*

According Duménil's comment, my demonstration of the "marxist transformation theorem" (part III-2 of my paper), which "embodies [my] own contribution," is redundant given his own contribution [3]. In fact, he thinks that the equality "sum of profits = sum of surplus-values" is now secured "by definition," and that the theorem of existence of a system of prices of production according to the viability of the producing system is not interesting.

I agree that this theorem is very simple. Marx stated this result as an outcome of the definition, without questioning its simplicity. Yet one century of false debates arose about its proof.

I did not intend to prove that the viability of the producing system was a "*necessary* condition." Working as an economist (not a mathematician), I take for granted that the existing capitalist system is "viable," and I proved that a system of production prices such as defined by Marx may exist. I proved also this "sufficient condition" (within the framework of the "new solution") in the case of fixed capital, rents (see [6]); the generalization to joint production is straightforward.

But I am surprised by Duménil's idea that "viability has to be shown as an outcome of a price effect." Global viability of a producing system is an outcome of social and technological history, within which the history of prices is of real but small importance (see [8]).

II. Answer to P. Flaschel

The comment of Flaschel is also two-fold.

1. *The "Marxist Transformation Theorem" Is a Well-Known Sraffaian Theorem*

To my knowledge Sraffa never dealt with the Marxian transformation problem. In fact, Flaschel claims that

(a) Point 1 of Theorem III-2 is already demonstrated within Sraffa's framework.

I must point out that Sraffa's equation equivalent to my

$$(H_2){:}\ p = (1 + r)(pA + w\ell) \qquad \text{is} \qquad (H_3){:}\ p = (1 + r)\,pA + w\ell.$$

It is well known that $H_1 + H_3$ admits one solution, but since $H_1 + H_2$ needs a 6-line demonstration, I thought it would be more convenient to give it directly.

(b) Point 2 (the "marxian equalities") is straightforward if one considers "values" as Sraffa's prices with $r = 0$.

The straightforwardness is already in my demonstration. But it is questionable that values are Sraffa's prices with $r = 0$. In fact, as Benetti and Cartelier [2] have pointed out, the data "ℓ" in Sraffa are nothing but a description of the distribution of the surplus, not a measure of abstract labor. Between abstract labor and price of labor-power there is a linear transformation (the tensor of exploitation T of my paper), which can be omitted formally, but not in an economic interpretation.

2. *What to Do with the Transformation*

According to Flaschel, the main use of the transformation is to make explicit the divergence between the "value rate or profit" and the "price rate of profit," this divergence being subject to empirical investigations.

This may be an interesting research program, though I suspect that the errors in the data could be greater than the divergence itself.

In my opinion, the purpose of the "transformation" is theoretical. It establishes a link between the "inner" aspects of the capitalist economy (productivity, and the rate of exploitation) and the "external" aspects (prices, revenues). Thus the transformation is the first step of a marxist theory of the world of *nominal revenues*: the "enchanted world" that Marx intended to analyse in his uncompleted Volume III (see [9]).

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