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The "Transformation Problem"

The definition of the " value " of a commodity as the (socially necessary) labour-time expended on its production is the cornerstone of the Marxian doctrine of capitalist exploitation. Yet Marx was well aware that the society of his day did not in fact exchange commodities in proportion to their " values ". His explanation for this departure from the " simple law of value " was the presumed tendency of capitalists to shift their resources from one industry to another until the resulting scarcity relationships had established a system of commodity prices which equalized the rate of profit (on cost) in all branches. In this manner labour values were said to be " transformed " into the Marxian prices of production.[1]

The arithmetic illustration of the transformation process which Marx gave in Vol. III of *Capital*[2] has been the subject of a long drawn-out controversy. Bohm-Bawerk, one of the first to call attention to the obvious inadequancies of the exercise, was generally taken to imply that the transformation of " values " into prices[3] as conceived by Marx was a logical impossibility. Since then a number of authors have come to the defence of Marx with attempts to demonstrate the internal consistency and determinacy of his conception by means of an algebraic treatment of the problem. The latest contribution was made by Mr. Meek in a recent issue the Economic Journal,[4] to which the reader may be referred for a useful bibliography of the subject.

It appears to me that a rigorous analysis of the transformation problem should pursue two distinct objectives : Firstly, it should establish whether, and under what conditions, the problem admits of a uniquely determined solution ; and, secondly, it ought to reveal whether or not this solution possesses certain characteristics which Marx had used in the further development of his system.[5] The three best known post-Marxian solutions have concentrated on the first objective without explicit reference to the second. This article attempts to deal with both aspects of the problem in turn.

THE PRINCIPLE OF EQUAL PROFITABILITY

The proof of the general consistency and determinacy of the problem has often been described as mathematically trivial. Yet few things can be as obscure and easily misunderstood as mathematical trivialities when they involve economic relationships, and most writers have unwittingly concealed the trivial nature of their solution by seeming to make

[1] It should be noted that in the Marxian system the " prices of production " (defined as cost *plus* profit at the average rate) are only the first approximations to actual *market prices.*

[2] *Capital*, Vol. III (Kerr edn.), pp. 182-203.

[3] i.e., " prices of production ".

[4] Ronald Meek, " Some Notes on the ' Transformation Problem ' " (*Economic Journal*, March 1956, p. 94).

[5] More particularly, I have here in mind the proposition that prices will exceed " values " in those branches of production where the " organic composition of capital " falls below the national average, and conversely.

it dependent on unnecessarily restrictive assumptions. Foremost among these has been the subdivision of the economy into three " departments " producing capital goods, wage goods, and luxury goods respectively, with the corollary that every physical commodity was not merely unequivocably identifiable as the product of one or other of these, but that its ultimate use in the economy was equally invariable, and predetermined by its department of origin : Capital goods were only " consumed " by factories, wage goods by workers, and luxury goods by capitalists. It can be shown, however, that the most general n-fold subdivision of the economy, in which each product may be distributed among *several* or *all* possible uses is equally acceptable—and easily handled—as a premiss for the required proof.[1]

Let k_{ij} represent the " cost input " of industry j's product into industry i (reckoned in terms of labour value)—where the term " cost input " is taken to cover the portion used for further processing (the usual *technological* connotation of " input ") and the quantity bought out of wages by the workers of industry i for their own consumption. In other words k_{ij} comprises both " machine feeding " and " labour feeding " input, and the only element excluded from its purview is the portion of industry j's output which is consumed by capitalists or used for investment purposes. The allocation of this portion (e_j) among consuming industries will not be specified in our model. The structure of the economy can then be represented by a scheme closely allied to the familiar Leontief matrix :

$$
\begin{array}{lllllllllll}
k_{11} & + & k_{21} & + & \ldots & + & k_{n1} & + & e_1 & = & a_1 \\
k_{12} & + & k_{22} & + & \ldots & + & k_{n2} & + & e_2 & = & a_2 \\
 & \cdot & \cdot & & \cdot\;\cdot\;\cdot & & \cdot & \cdot & \cdot & & \\
 & \cdot & \cdot & & \cdot\;\cdot\;\cdot & & \cdot & \cdot & \cdot & & \\
k_{1n} & + & k_{2n} & + & \ldots & + & k_{nn} & + & e_n & = & a_n \\
s_1 & + & s_2 & + & \ldots & + & s_n & & & = & s
\end{array} \tag{1}
$$

where a horizontal reading shows the allocation of each industry's output according to *destination*, and a column-wise reading the structure of each industry's " cost-input " according to *origin* (including the residual " surplus " s accruing to it.) The sum of each column must of course be equal to the sum of the corresponding row.[2]

[1] Mr. K. May (*Economic Journal*, December 1948) has preceded me in pointing to the hidden generality of the traditional solutions. It is not clear from his remark, however, whether he merely believed that the *number* of departments could be indefinitely increased or whether he was aware that, in addition, the postulate of invariable use could also be relaxed.

[2] If the k_s of any column (i) were resolved into their constituent portions of technological inputs (c_{ij}) and labour-feeding inputs (v_{ij}) and these were separately summed over all industries of origin (j), the column would reproduce the familiar Marxian value equation : $a_i = c_i + v_i + s_i$, where c_i and v_i stand for " constant " and " variable " capital respectively. There are various qualifications to this, notably the existence of a state of " simple reproduction " in which the e's are wholly absorbed by capitalists' consumption and do not contain an investment element ; for the Marxian type of process analysis does not allow the output flows of any period to serve as the input flows of the *same* period, but holds them over for consumption in the *next*. Unless, therefore, each period was exactly like the previous one in all respects (" simple reproduction ") the input structure appropriate to the Marxian value equation could not be deduced from the output distribution of the same period (i.e., $k_{ij} \neq c_{ij} + v_{ij}$).

It is now quite easy to show how this system of " *value* " flows can be uniquely translated into *price* terms. If p_i is the price of industry i's product (per unit of labour value), the requirement of equal profit ratios (π) in all industries[1] may be expressed as follows :

$$
\begin{array}{lcccl}
k_{11}\,p_1 + k_{12}\,p_2 & \ldots & + \; k_{1n}\,p_n & = & \rho a_1\,p_1 \\
k_{21}\,p_1 + k_{22}\,p_2 & \ldots & + \; k_{2n}\,p_n & = & \rho a_2\,p_2 \\
\cdot \quad \cdot & \cdot\;\cdot\;\cdot & \cdot & & \cdot \\
\cdot \quad \cdot & \cdot\;\cdot\;\cdot & \cdot & & \cdot \\
k_{n1}\,p_1 + k_{n2}\,p_2 & \ldots & + \; k_{nu}\,p_n & = & \rho a_n\,p_n
\end{array}
\tag{2}
$$

where ρ stands for the equalized " cost ratio " ($\rho = 1 - \pi$). Dividing each equation by the relevant total output a_i ,and defining the cost-input coefficients $\kappa_{ij} \equiv k_{ij}/a_i$, this may be written :

$$
\begin{array}{lllll}
(\kappa_{11}-\rho)p_1 & + \; \kappa_{12}\,p_2 & \ldots & + \; \kappa_{1u}\,p_n & = 0 \\
\kappa_{21}\,p_1 & + \; (\kappa_{22}-\rho)p_2 & \ldots & + \; \kappa_{2n}\,p_u & = 0 \\
\cdot \quad \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \quad \cdot & \cdot \\
\cdot \quad \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \quad \cdot & \cdot \\
k_{n1}\,p_1 & + \; k_{u2}\,p_2 & \ldots & + \; (k_{nn}-\rho)p_n & = 0
\end{array}
\tag{3}
$$

We are thus provided with n homogeneous equations in n unknowns ($p_1, p_2 \ldots p_n$), whose consistency—according to a fundamental theorem in algebra—requires the vanishing of their determinant, i.e. :

$$
\begin{vmatrix}
\kappa_{11}-\rho & \kappa_{12} & \ldots & \kappa_{1n} \\
\kappa_{21} & \kappa_{22}-\rho & \ldots & \kappa_{2n} \\
\cdot & \cdot & \cdot\;\cdot & \cdot \\
\cdot & \cdot & \cdot\;\cdot & \cdot \\
\kappa_{n1} & \kappa_{n2} & \ldots & \kappa_{nn}-\rho
\end{vmatrix}
\equiv \; | k - \rho I | \; = 0
\tag{4}
$$

The consistency condition (4) determines the average cost ratio ρ (and hence the profit ratio π) as a function of the (known) input coefficients in value terms (κ_{ij}).[2] A number of simple propositions concerning ρ, most of which are intuitively obvious may be deduced from the mathematical form of this condition, but need not detain us unduly.[3]

[1] For algebraic convenience we define the " profit ratio " π as the ratio of profit to *total value of output*. Obviously π will be equal in all industries if and only if the Marxian " rate of profit " (profit $\div$ total *cost*) is similarly equalized.

[2] In the Western literature on input-output schemes the matrix of technological input coefficients is often referred to as the " technology " of the economic system. The inclusion of " wage inputs " in the coefficients k_{ij} transforms this matrix into what might be called the " augmented technology " of the system. Having regard to equation (4) and using well-established algebraic terminology, the average cost ratio may therefore be described as a *latent root* of the " augmented technology ".

[3] The following are the most important characteristics of ρ which flow from its definition as a latent root of (4), (see footnote 3 above) :

(1) Although the κ_{ij} are expressed in terms of *labour values*, their replacement by the corresponding *physical* coefficients κ_{ij}^* would not alter the latent root ρ ; for it is evident that $\kappa_{ij}^* = \kappa_{ij}\,\lambda_i/\lambda_j$ (where the λs stand for the labour value of the physical units) and that the suggested replacement would merely result in a collineatory transformation of the matrix κ (i.e., $\kappa^* = \Lambda\kappa\,\Lambda^{-1}$ where Λ is the diagonal matrix of the λs). It is well known that such a transformation leaves the latent roots invariant.

(2) Since none of the elements of κ can be negative, we may deduce further characteristics of κ from Frobenius' theorems on the latent roots of positive motives (Sitzungsberichte der k. preussischen Akademie der Wisseuschaften, 1908, vol. I, p. 471. The fact that some of the elements of κ may vanish might necessitate slight modifications in very exceptional cases which will be neglected here ; see *ibid.* 1912, vol. I, p. 456): A real positive solution for ρ will always exist, and must lie between the smallest and the largest column-sum of κ, i.e., between the smallest and the largest *cost-ratio in value terms*. It follows that in the special case where this cost-ratio is equal in all industries (i.e., all column-sums of k are equal), it must also be equal to the *cost-ratio in price terms* ρ and consequently the average profit ratio will equal the ratio of surplus ($\Sigma s/\Sigma a$).

(3) As the (dominant) latent root of a positive matrix is a monotonically increasing function of each of the matrix elements (*ibid.*), the reduction of any input coefficient k_{ij} through technological progress or increased " exploitation of labour " will *cet. parbius* reduce the cost ratio and correspondingly increase the average rate of profit.

When our solution for ρ (in terms of the k_{ij}) is substituted in (3) the system will determine the n prices p_i but for a proportionality factor. *In other words we can obtain unique solutions for the relative prices in terms of any one commodity* (say n): $p_1/p_n, p_2/p_n, \ldots p_{n-1}/p_u$, and this is as far as the principle of equal profitability will take us.

POSTULATES OF INVARIANCE

In order to determine the *absolute* prices (as opposed to price *ratios*) a further as yet unspecified condition is required and this may be chosen from quite a variety of alternatives. Essentially what it amounts to is the selection of a definite aggregate (or other characteristic) of the value system (1) *which is to remain invariant to the transformation into prices.* The Marxian texts contain references and *obiter dicta* which could be made to support a number of mutually incompatible " invariants ", and it will be useful to pass in review the ways in which previous analysts of the transformation problem have differed in their selection.

The Bortkievicz-Sweezy analysis[1] claims invariance for the unit-value of luxury goods (the products of department III in the traditional three-sector analysis) i.e. :

$$p_3 = 1 \tag{5a}$$

The postulate is designed to ensure that prices will be expressed *in terms of the value of gold* (a product of department III) which brings the solution into line with Marxian monetary theory. A closely allied, though so far neglected, alternative might be the invariance of the unit value of *wage goods* (i.e. $p_2 = 1$ in the three-sector analysis) which would appear to be supported by the Marxian notion that even under capitalism " the worker is paid the full value of his labour " and exploitation (i.e. the withholding of the *surplus*) is concealed by " commodity fetishism ".

Other analysts have allowed *unit*-values to change, and preferred to claim invariance for value *aggregate.* Thus the Winternitz[2] approach is based on the Marxian dictum that " *total value equals total price* " i.e. :

$$\Sigma a = \Sigma ap. \tag{5b}$$

This postulate has of course the advantage of symmetry and claims no special position for any one of the three departments. It is no longer a *single* price but a weighted average of *all* prices that is equal to unity (i.e. equal to value). Immediate expression is therefore given to the Marxian theorem that some price will *exceed* values and others *fall short* of them—a proposition which the Bortkievicz postulate may in certain circumstances contradict.

[1] Originally advanced by Bortkievicz and later simplified by P. Sweezy in *Theory of Capital Development*, New York, 1942, pp. 109, *et. seq.*

[2] Values and Prices : " A Solution of the so-called Transformation Problem " (*Economic Journal*, June, 1948, p. 276).

As a third alternative, one might claim invariance for the *surplus* rather than for aggregate *output* and postulate the equality of total profit (in price terms) with total surplus (in value terms), as Mr. Meek has done,[1] i.e. :

$$(5c) \qquad \Sigma s = \Sigma s_p = (1 - \rho)\ \Sigma ap$$

This is consonant with the Marxian *facon de parler* that capitalists " redistribute the surplus " among themselves in proportion to their capital, a process which (if nothing else were involved) ought obviously to leave the sum-total of surplus unaffected.[2]

No doubt the three alternative postulates (5*a*), (5*b*) and (5*c*) do not exhaust all the possibilities. There may be other aggregates or relationships with perfectly reasonable claims to invariance whose candidacy has not so far been pressed. But the point which concerns us here is that *the principle of equal profitability* (2) *in conjunction with any one invariance postulate will completely determine all prices* ($p_1 \ldots p_n$)[3] and thereby solve the transformation problem. However, there does not seem to be an objective basis for choosing any particular invariance postulate in preference to all the others, *and to that extent the transformation problem may be said to fall short of complete determinacy.*

It should be noted at this point that some of the postulates advanced in recent years do not fulfil the essential function of determining absolute price levels, and may even be incompatible with the principle of equal profitability in all branches. Thus, both Mr. Dobb and Mr. Meek[4] have advocated a modification of (5*b*) which they believe to be more in the spirit of Marxism and postulated the equality of total value with total price *in terms of wage goods* (the products of department II), i.e. :

$$(6a) \qquad \Sigma a = \frac{\Sigma ap}{p_2} \quad \left(or\ \frac{\Sigma a}{\Sigma v} = \frac{\Sigma ap}{p_2 \Sigma v} = \frac{\Sigma ap}{\Sigma vp}\right)$$

As may be seen from the bracketed version, this is tantamount to the invariance of the output ÷ wages ratio.[5] It is clear, however, that (6*a*) says nothing about *absolute* prices ; it merely imposes an additional, and supernumerary, condition on the relative prices (p_1/p_2, $p_3/p_2 \ldots$) *which are already determined by the principle of equal profitability.* Unless, therefore, our basic model (1) obeys certain well-defined mathematical constraints, we cannot postulate (6*a*) alongside that principle. The same will of course be true of any invariance postulate which involves only price *ratios*, and this debars us equally from claiming invariance for the output — surplus ratio[6] :

$$(6b) \qquad \frac{\Sigma a}{\Sigma s} = \frac{\Sigma ap}{\Sigma sp_3} = \frac{\Sigma ap}{p_3\ \Sigma s}\left(or\ \Sigma a = \frac{\Sigma ap}{p_3}\right),$$

unless we allow the basic model to depart from generality in a definite manner.

[1] Roland Meek, *op. cit.*

[2] In the traditional three-sector analysis and *under conditions of simple reproduction*, the postulate (5*c*) is equivalent to the Bortkievicz postulate (5*a*), since the " surplus " will then consist exclusively of department III products, i.e., luxuries for capitalists' consumption.

[3] If the matrix has several positive latent roots the solution will not be unique, but this eventuality can, I think, be safely neglected.

[4] *Ibid.*

[5] I am not sure if I can follow Mr. Dobb and Mr. Meek in their insistence that such a postulate is particularly well-founded in Marxian doctrine. Why should we require the invariance of the output ÷ wages ratio rather than that of, say, the surplus ÷ wages ratio ? If Marx regarded the " degree of exploitation " as the critical magnitude in any given capitalist economy, then, surely, it is the relation between the *incomes* of social classes which ought to survive the transformation, rather than that between wages and aggregate *output*.

[6] For simplicity the postulate is here presented in the form appropriate to " simple reproduction " with three departments.

THE VALUE MODEL UNDER SPECIAL ASSUMPTIONS

In our endeavour to assess the degree of determinacy of the transformation problem we have so far based ourselves on the most general model of value flows amenable to mathematical treatment (equations 1). This generality must be abandoned when we wish to investigate the *characteristics* of the Marxian prices of production as opposed to their *uniqueness* or *determinacy*. We shall therefore begin by recasting our basic scheme into the special Marxian mould of simplifying assumptions. As a first step the n industries will be reduced to the familiar three departments (I = producer goods used in further processing, II = wage goods consumed by workers, III = luxury goods consumed by capitalists), and we shall simplify our notation by writing c_i for k_{i1} and v_i for k_{i2}. The k_{i3}s must all vanish since luxury goods do not function as " cost-inputs " (either technological or " labour-feeding ") and equations (1) will therefore reduce to :

$$
\begin{array}{lllllll}
c_1 & + c_2 & + c_3 & + e_1 & = a_1 \\
v_1 & + v_2 & + v_3 & + e_2 & = a_2 \\
. & . & . & + e_3 & = a_3 \\
s_1 & + s_2 & + s_3 & & = s
\end{array} \tag{7}
$$

The columns are now an explicit statement of the Marxian value equations[1] $a_i = c_i + v_i + s_i$ i.e. *total value* = *constant capital* + *variable capital* + *surplus* and certain key concepts of the Marxian system can easily be defined for each of three departments :

$$\omega_i \equiv \text{organic composition of capital} = \frac{c_i}{c_i + v_i}$$

$$\varepsilon_i \equiv \text{rate of exploitation} = s_i/v_i$$

The principle of equal profitability (2) will now simplify to :

$$
\begin{aligned}
c_1 p_1 + v_1 p_2 &= \rho a_1 p_1 \\
c_2 p_1 + v_2 p_2 &= \rho a_2 p_2 \\
c_3 p_1 + v_3 p_2 &= \rho a_3 p_3
\end{aligned} \tag{8}
$$

A unique solution for relative prices can easily be obtained by the previous method since (8) may be written as :

$$
\begin{aligned}
(\gamma_1 - \rho)p_1 + \upsilon_1 p_2 \quad + \quad . &= 0 \\
\gamma_2 p_1 + (\upsilon_2 - \rho) p_2 + \quad . &= 0 \\
\gamma_3 p_1 + \upsilon_3 p_2 \quad - \rho p_3 &= 0
\end{aligned} \tag{9}
$$

where γ_i and υ_i stand for the " *constant*—" and " *variable*—" capital ratios respectively ($\gamma_i \equiv c_i/a_i$ and $\upsilon_i \equiv v_i/a_i$). The latter are of course no more than simplifications of the general " cost-input " coefficients κ_{ij}, with the technological and the " labour-feeding " elements neatly separated thanks to the particular delineation of the three industries. It is only by virtue of this separation that the " organic compositions of capital " and " rates of exploitation " enter into the determination of production prices at all and that the Marxian assumptions concerning their role in the transformation process can be analytically tested.

[1] Strictly speaking this is only true in conditions of " simple reproduction ", i.e., when $e_1 = e_2 = 0$ see footnote 2 on page 149). Under " expanded reproduction " equations (7) cannot be interpreted in terms of " value components " at all, but this does not affect the conclusions of this section, since conditions 8) and all subsequent equations remain valid in any case.

As in the general case (see 4) the consistency of equations (9) requires the vanishing of their determinant :[1]

$$(10)\qquad 0 = \begin{vmatrix} \gamma_1-\rho & \upsilon_1 & 0 \\ \gamma_2 & \upsilon_2-\rho & 0 \\ \gamma_3 & \upsilon_3 & -\rho \end{vmatrix} = -\rho \begin{vmatrix} \gamma_1-\rho & \upsilon_1 \\ \gamma_2 & \upsilon_2-\rho \end{vmatrix}$$

This furnishes a solution for the average cost ratio ρ, which may be substituted in (9) to make the system uniquely solvable for the three prices, except for the familiar proportionality factor. The latter, of course, can only be supplied by one or other of a possible range of invariance postulates (such as 5*a*, 5*b*, or 5*c*).

Since, however, there is no objective criterion of selection between these postulates, it might be desirable to look for special assumptions concerning the value system (7) which would make *several* or *all* of them compatible at one and the same time. A specialized model of this sort, if it could plausibly be accepted, would remove the last remaining element of indeterminacy from the transformation problem. Several simple possibilities spring to mind :

(1) Mr. Meek assumes that the organic composition of capital in the wage goods industry is equal to the national average i.e. $c_2/(c_2 + v_2) = \Sigma c/(\Sigma c + \Sigma v)$. He also retains the usual Marxian assumption of equal rates of exploitation in all departments which, in addition, implies $s_2/v_2 = \Sigma s/\Sigma v$. In this way Department II becomes a simple scale model of the total economy (c_2: v_2: $a_2 = \Sigma c$: Σv: Σa), and we can replace the value components of the second equation in (8) by the corresponding *total* aggregates without affecting its validity i.e. :

$$(\Sigma c)p_1 + (\Sigma v)p_2 = \rho(\Sigma a)p_2$$

However, as a simple summation of the three equations (8) will show, the left-hand side above must also equal $\rho\Sigma ap$ and it follows that $(\Sigma a)p_2 = \Sigma ap$ or :

$$(11)\qquad \frac{\Sigma a}{\Sigma v} = \frac{\Sigma ap}{p_2\,\Sigma v} = \frac{\Sigma ap}{\Sigma vp}$$

Thus, the invariance of the output ÷ wages ratio which we have rejected as an independent postulate in the general case has now been shown to hold *necessarily* when the value structure of the wage goods industry conforms to the national average. I am not sure, however, that such a radical departure from generality is not too high a price to pay for the rather doubtful orthodoxy which an invariant output ÷ wages ratio would impart to

[1] An interesting characteristic of the three department assumption is the fact that industry III is by definition incapable of contributing " cost-inputs " to the other two. It follows that its value components cannot enter into the determination of the prices and profit-ratios of departments I and II and the latter must find their level *regardless of the structure of department III.* Once this has happened, however, the profit ratio established in I and II must spread to department III also (since the Marxian equilibrium demands equal profitability everywhere), and this will determine the relative price of luxury goods (given the capital ratios γ_3 and υ_3).

Mathematically, these propositions follow from the two zeros in the last column of the determinant and its consequent proportionality to a *two*-rowed determinant (R.H.S. of 10). Since the proportionality factor cannot be zero, the latter must necessarily vanish, thus giving a solution for ρ dependent on the structure of the first two departments only. When this is substituted in the first two equations of (9), the price ratio p_1/p_2 can be determined independently of department III. Although in the particular case of " simple reproduction " this independence is destroyed by the functional relationship between the three departments (as Mr. May has pointed out), it seems to me real enough in the general case of " expanded reproduction ".

the model[1], particularly since the determination of the absolute price level is still not achieved and requires a further postulate, such as (5*b*) or Mr. Meek's own (5*c*). The only advantage that might conceivably accrue would be the possibility of simultaneous invariance for the aggregate output value (5*b*) and the unit value of wage goods ($p_2 = 1$ a variant of (5*a*)). But Mr. Meek does not exploit this possibility.[2]

(2) Suppose now that it was the *capital goods* industry (department I), rather than the wage goods industry, which was to be a scale model of the whole economy ($c_1 : v_1 : a_1 = \Sigma c : \Sigma v : \Sigma a$). In that case we can write :

$$(\Sigma c)p_1 + (\Sigma v)p_2 = \rho(\Sigma a)p_1 \text{ (by virtue of the first equation of (8))},$$
$$\text{and} \quad (\Sigma c)p_1 + (\Sigma v)p_2 = \rho \Sigma ap \text{ (by simple summation of (8))}.$$

Equating the two right-hand sides and dividing by $\rho p_1(\Sigma c)$, we obtain :

$$\text{(12)} \qquad \frac{\Sigma a}{\Sigma c} = \frac{\Sigma ap}{p_1 \Sigma c} = \frac{\Sigma ap}{\Sigma cp}$$

Thus, the " representativeness " of department I is seen to imply the invariance of the output — constant-capital ratio.

(3) By an exactly analogous process it can be shown that the assumption of representativeness for the *luxury* industry (i.e., $c_3 : v_3 : a_3 = \Sigma c : \Sigma v : \Sigma a$) would imply the equality $\Sigma a / \Sigma s = \Sigma ap / (\Sigma s)p_3$. In this case, however, we cannot take the further step of deducing any meaningful invariance unless we make the additional assumption of " simple reproduction " (i.e., $\Sigma s = e_3$ and therefore $(\Sigma s)p_3 = \Sigma sp$). If this obtains, a " representative " luxury industry will imply the invariance of the output $\div$ surplus ratio :

$$\text{(13)} \qquad \frac{\Sigma a}{\Sigma s} = \frac{\Sigma ap}{\Sigma sp}$$

In some ways this might be the most satisfactory model of all, as it would enable us to postulate *all three* invariances (5*a*, 5*b*, and 5*c*) at one and the same time. We could allow " total price " to equal " total value ", speak of a fixed fund of surplus being " redistributed among capitalists in proportion to their capital ", and at the same time permit money prices to be expressed in terms of the value of gold. The model could thus impart complete determinacy to the transformation problem while satisfying all the Marxian preconceptions as to the characteristics of the solution. It is, however, a very restrictive model and may not commend itself in view of its radical departure from generality.[3]

[1] As Mr. Meek correctly points out, the proposed departure from generality is only a *sufficient* and not a *necessary* condition of this invariance. However, the alternative assumptions which might establish it, have no recognizable economic meaning other than the postulation of the invariance itself.

[2] It could of course be argued that the postulation of 5(*b*) on top of the other assumptions would have brought his model altogether too near triviality. Incidentally, while Mr. Meek is perfectly entitled to his choice of figures departing from " simple reproduction ", it is a little confusing to find that his wage goods industry is contracting, while the other two departments expand. There is nothing *logically* inconsistent in this, but it does seem unnecessarily odd.

[3] As a simple illustration of such a model, we suggest the following figures :

		c		v		s		a
	Dept. I	80	+	20	+	20	=	120
Value-system :	Dept. II	10	+	25	+	25	=	60
	Dept. III	30	+	15	+	15	=	60

The transformation (under the assumptions specified) yields the prices $p_1 = \frac{6}{5}$, $p_2 = \frac{3}{5}$, $p_3 = 1$ and we therefore have :

		cp		vp		sp		ap
	Dept. I	96	+	12	+	36	=	144
Price system	Dept. II	12	+	15	+	9	=	36
	Dept. III	36	+	9	+	15	=	60

The profit ratio (25%) is now equalized in all departments and both aggregate value (240) and total surplus (60) have remained unchanged. Apart from " simple reproduction " (*i*th row-sum = *i*th column-sum), this result is made possible by the identical value structure of Dept. III and the total economy (30 : 15 : 15 = 120 : 60 : 60).

THE DEVIATION OF PRICES FROM VALUES

No analysis of the transformation problem is entirely satisfactory unless it throws some light on the important Marxian assertion that prices will exceed values ($p_i > 1$) in industries with a higher than average " organic composition of capital " (non-wage share in total capital), and fall short of them in branches with the opposite characteristic. The importance of this theorem to Marxist ideology, particularly in its newest Soviet setting, derives from its alleged implications concerning the process of industrialization under capitalism and socialism respectively. To the Marxist way of thinking, as Mr. Meek has pointed out, the transformation of values into prices is not merely a *logical*, but also a *historical* progress. Thus, in the early stages of capitalism, when this transformation has hardly begun, the rate of profit obtainable in capital goods industries (whose " organic composition " is held to be relatively high) will not as yet have reached equality with that of consumer goods industries.[1] Capitalists will therefore prefer to invest their resources in the latter until the transformation has gone far enough to equalize the rate of profit everywhere. In Marxist ideology, therefore, the process of capitalist industrialization is bound to begin with the development of light industry (textiles, sugar, etc.), and to delay the take-off of heavy industry (metals, engineering, etc.) until a comparatively advanced stage has been reached. This is held to be an obstacle to the realisation of the fastest rate of growth attainable on technological grounds, and to discourage the fullest use of labour-saving methods even when capitalism has reached maturity (owing to the inevitable " over-pricing " of means of production). Thus, in the Marxist view, society is cheated of the fruits of technological advance by the capitalist requirement of equal profitability, and the claims of socialism as a speedier engine of industrialization and greater liberator from human toil can be more plausibly advanced to the extent that it can dispense with this requirement and start the process from the opposite end of heavy industry.

While it would be out of place to enter into the metaphysics, or even the logic, of this argument, it is obviously desirable that we should test the validity of its premiss in the context of the transformation problem. At first sight the truth of the Marxian theorem may seem fairly obvious, particularly when we recall that it is the function of prices so to re-value each commodity that an initially equalized surplus ÷ *wages* ratio (" rate of exploitation ") is replaced by a universally valid surplus ÷ *total cost* ratio, — a process which would seem to require over-valuation wherever the wage component in total cost is relatively small (i.e., organic composition of capital is high), so that producers can be, as it were, compensated by the price system for the smaller proportion of resources which they can directly apply to extracting surplus from labour. Further reflection, however, will show that the conclusion is only obvious if we neglect the effect of the universal price transformation on the *cost structure* of each industry, as Marx had done in his famous (and inconclusive) arithmetical example. As soon as input effects are taken into account, as they surely must be for complete consistence, the Marxian theorem is far from obvious and requires special proof. The issue has, I believe, been shirked by those analysts who attempted explicit algebraic solutions for the prices of production, and who were evidently deterred by the extreme complexity of the mathematical expressions emerging in the process. Yet the clear recognition of important features of a solution would often be more desirable than its explicit rendering, particularly when the latter is bound to be so cumbersome as to defeat the very object of explicitness. Fortunately it is possible to give the required proof without recourse to an explicit solution.

[1] The " rate of expliotation " being assumed equal everywhere.

The principle of equal profitability (8) obviously requires that :

$$\frac{c_i p_1 + v_i p_2}{a_i p_i} = \frac{(\Sigma c)p_1 + (\Sigma v)p_2}{\Sigma ap}$$

for all i (= 1, 2, 3). If, in addition, we postulate that " total price equals total value ", i.e., $\Sigma ap = \Sigma a$, the condition reduces to :

$$\frac{\gamma_i p_1 + \upsilon_i p_2}{p_i} = \gamma_0 p_1 + \upsilon_0 p_2 \tag{14}$$

where the γs and υs are the constant- and variable-capital ratios of the departments (c_i/a_i and v_i/a_i) and of the economy as a whole ($\gamma_0 \equiv \Sigma c/\Sigma a$ and $\upsilon_0 \equiv \Sigma v/\Sigma a$), and it follows at once that any absolute price p_i can be expressed in terms of the single price ratio ($p_1/p_2 \equiv p$) :

$$p_i = \frac{\gamma_i p + \upsilon_i}{\gamma_0 p + \upsilon_0} \tag{15}$$

Now it is clear that the Marxian assumption of an equal " rate of exploitation " implies a definite dependence between the γs and the υs. For if the ratio of capitalists' to workers' incomes is equal everywhere, so is the ratio of workers' to *total* income λ $\left(\equiv \frac{v_i}{a_i - c_i} = \frac{v_i}{1 - \gamma_i}\right)$, and it follows by substitution in (15) that :

$$p_i = \frac{\gamma_i(p - \lambda) + \lambda}{\gamma_0(p - \lambda) + \lambda} \tag{16}$$

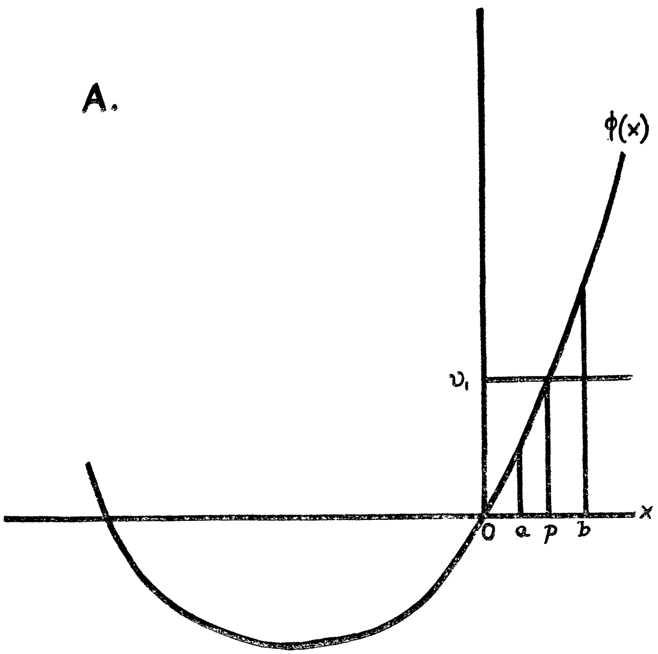
It is obvious, therefore, that *provided* $(p - \lambda)$ *can be taken to be positive*, prices will exceed values ($p_i > 1$) if, and only if, the capital ratio γ_i exceeds the national average γ_0. But since λ is equal in all branches, this can only be so if the organic composition of capital $\left(\omega_i = \frac{\gamma_i}{\gamma_i + \upsilon_i}\right)$ is also in excess of the national average. The converse ($p_i < 1$) will of course apply wherever γ_i falls short of γ_0 (and therefore ω_i of ω_0).

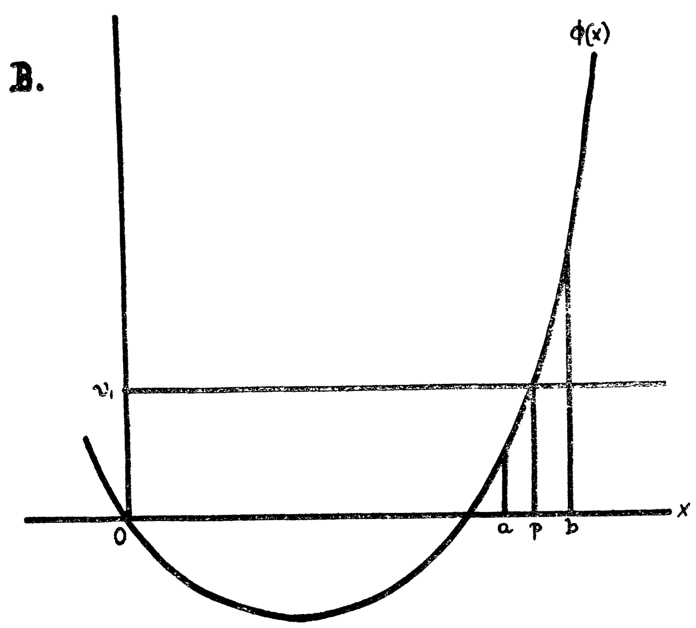
Hence, all that remains to be done in order to prove the truth of the Marxian theorem is to establish that $p - \lambda$ will of necessity be positive, i.e., that the price ratio of the first two departments (p_1/p_2) must exceed the equalized wage ratio λ. This somewhat surprising lemma may be demonstrated as follows :

Since the L.H.S. of (14) remains equal, whether i = 1 or 2, we must have :

$$\phi(p) = \gamma_2 p^2 - (\gamma_1 - \upsilon_2)p = \upsilon_1 \tag{17}$$

Let us now reproduce the geometric shape of the function $\phi(x)$ for the alternative cases A $(\gamma_1 < \upsilon_2)$ and B $(\gamma_1 > \upsilon_2)$:





In either case it is easily seen that whenever (for a positive a) $\phi(p)$ is larger than $\phi(a)$, p must itself be larger than a, and conversely in the case of b ($>p$). It follows, in particular, that p will exceed or fall short of λ, depending on whether $\phi(p)$ is larger or smaller than $\phi(\lambda)$. But $\phi(p)$ must *always* exceed $\phi(\lambda)$; for :

$$\phi(p) = \upsilon_1 = \lambda(1 - \gamma_1)$$

and

$$\phi(\lambda) = \gamma_2\lambda^2 - (\gamma_1 - \upsilon_2)\lambda = \lambda(\lambda - \gamma_1)$$

Since $1 > \lambda$, we must have $\phi(p) > \phi(\lambda)$ and therefore $p > \lambda$.[1] Thus $p - \lambda$ is always positive and the Marxian theorem clearly follows from (16), i.e. :

(20) $$p_1 \gtreqless 1, \textit{ if } \gamma_i \gtreqless \gamma_0, \text{ i.e., } \textit{if } \omega_i \gtreqless \omega_0.$$

In concluding this article it is essential to enter an important *caveat*. While the internal consistency and determinacy of Marx's conception of the transformation process, and the formal inferences he drew from it, have been fully vindicated by this analysis, the same can certainly not be said of the body of the underlying doctrine, without which the whole problem loses much of its substance and *raison d'etre*. The assumption of equal " rates of exploitation " in all departments has never to my knowledge been justified. Neither has the notion that the " organic composition of capital " must needs be higher in the capital goods industries than elsewhere in the economy. Above all, the denial of productive factor contributions other than those of labour, on which the whole doctrine of the surplus rests, is an act of *fiat* rather than of genuine cognition. It is these doctrinal preconceptions which must remain the centre of any reappraisal of Marxian economics, rather than the logical superstructure which our analysis has shown to be sound enough.

Oxford. F. SETON.

[1] I am indebted to Professor H. G. Johnson for an alternative proof which is not dependent on visual aids, and *pro tanto* more rigorous :

Equation (17) must have one positive and one negative root (since their product is $-\upsilon_1/\gamma_2$), of which only the former is economically relevant. We can therefore write :

$$p = \frac{\gamma_1 - \upsilon_2}{2\gamma_2} + \sqrt{\left(\frac{\gamma_1 - \upsilon_2}{2\gamma_2}\right)^2 + \frac{\upsilon_1}{\gamma_2}}$$

By virtue of λ being equal to $\upsilon_i/(1 - \gamma_i)$, this transforms into :

$$p - \lambda = \left(\frac{\gamma_1 - \upsilon_2}{2\gamma_2} - \lambda\right) + \sqrt{\left(\frac{\gamma_1 - \upsilon_2}{2\gamma_2} - \lambda\right)^2 + \frac{\lambda(1 - \lambda)}{\gamma_2}}$$

It follows at once that as long as $\lambda < 1$, $p - \lambda$ must be positive.